Matrices

- A **matrix** can be thought of as an array of numbers (a collection of numbers set out in a table) and they come in different shapes and sizes.
- You can describe these different shapes and sizes in terms of the **dimension** of the matrix. This is given by two numbers n and m in the form $n \times m$ (read as n by m), where n is the number of rows (horizontal or across the page) and m is the number of columns (vertical or down the page) in the matrix.
- An $n \times m$ matrix has n rows and m columns.
- Matrices are usually denoted in bold print with a capital letter e.g. **A**, **M** etc.

Example

Solution

Give the dimensions of the following matrices

a $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, **b** $\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$,

c $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$, **d** $\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix}$.

You can add and subtract matrices of the same dimension.

Example

Solution

Find **a** $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 5 & 3 \end{pmatrix}$

b $\begin{pmatrix} 1 & -3 & 4 \\ 2 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 1 \\ 5 & 2 & 3 \end{pmatrix}$

- **To multiply a matrix by a number you simply multiply each element of the matrix by that number.**

Example

Solution

$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & 0 & -4 \end{pmatrix}$

Find **a** $2\mathbf{A}$ **b** $\frac{1}{2}\mathbf{B}$

Multiplying Matrices

- The basic operation consists of multiplying each element in the **row** of the left hand matrix by each corresponding element in the **column** of the right hand matrix and adding the results together.
- The number of columns in the left hand matrix must equal the number of rows in the right hand matrix.
- The product will then have the same number of rows as the left hand matrix and the same number of columns as the right hand matrix.

***Matrices can only be multiplied if the no. of columns of 1st matrix is equal to the no. of rows of the 2nd.**

***Matrices which can be multiplied are said to be conformable.**

So if

$\mathbf{A} \times \mathbf{B} = \mathbf{C}$

Dimensions: $(n \times m) \times (m \times k) \quad (n \times k)$

n is from the number of rows in **A**.
k is from the number of columns in **B**.

These numbers must be the same.

Example

Given that $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}$ find

a **AB**

b **BA**.

Solution

*Matrix multiplication is not Commutative i.e. AB≠BA

The Identity Matrix

The 2X2 Identity Matrix is **I**=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The 3X3 Identity Matrix is **I**=$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Where AI=IA=A *so multiplying a matrix BY I is a bit like multiplying a number by 1.

Inverse of a 2X2

- If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

  and then $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

The value of $ad - bc$ is called the **determinant** of **A** and written det(**A**).

- $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(\mathbf{A}) = ad - bc$ so $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Notice that if $\det(\mathbf{A}) = 0$ you will not be able to find $\mathbf{A}^{-1}$ because $\frac{1}{\det(\mathbf{A})}$ is not defined, in such cases we say **A** is **singular**.

- If $\det(\mathbf{A}) = 0$, then **A** is a **singular matrix** and $\mathbf{A}^{-1}$ cannot be found.
  If $\det(\mathbf{A}) \neq 0$, then **A** is a **non-singular matrix** and $\mathbf{A}^{-1}$ exists.

Transpose

- Given a matrix **A**, you form the **transpose** of the matrix $\mathbf{A}^{\mathrm{T}}$, by interchanging the rows and the columns of **A**. You take the first row of **A** and write it as the first column of $\mathbf{A}^{\mathrm{T}}$, you take the second row of **A** and write it as the second column of $\mathbf{A}^{\mathrm{T}}$, and so on.

  If $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 0 & -3 \\ 1 & 5 \end{pmatrix}$, then $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -3 & 5 \end{pmatrix}$.

- The transpose of a matrix of dimension $n \times m$ is a matrix of dimension $m \times n$.

- The transpose of a square matrix is another square matrix with the same dimensions. For example, the transpose of a 2×2 matrix is another 2×2 matrix.

- If $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, the matrix **A** is **symmetric**.

Example

Use an inverse matrix to solve the simultaneous equations

$$4x - y = 1$$

$$-2x + 3y = 12$$

Solution

*UPM p.161 Ex6A Q4,6-10

Transformations

A transformation of a plane takes any point A in the plane and maps it onto one and only one image A$'$ lying in the plane. We say that the point A(x,y) position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ has an image A'(x',y') position vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ under the transformation.

A transformation is said to be 'linear' if any linear relationship between position vectors is conserved under the transformation i.e. if a linear transformation maps a point A position vector $\boldsymbol{a}$ onto its image A' position vector $\boldsymbol{a'}$ and if $\boldsymbol{a} = \mu\boldsymbol{p} + \alpha\boldsymbol{q}$ then $\boldsymbol{a}' = \mu\boldsymbol{p}' + \alpha\boldsymbol{q'}$ where $\boldsymbol{p}'$ **and** $\boldsymbol{q}'$ are the images of $\boldsymbol{p}$ *and* $\boldsymbol{q}$.

All linear transformations of the plane can be expressed as a pair of equations of the form:-

$$\boldsymbol{x}' = ax + by$$

$$\boldsymbol{y}' = cx + dy$$

Or writing this in matrix form

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

Note

1. The transformation matrix can be found by finding the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ because
   $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$
   i.e. the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the first column of the transformation matrix
   and the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the second column of the transformation matrix.

2. Under any linear transformation the origin (0,0) maps to itself.
3. Under a translation by the vector $\begin{pmatrix} r \\ s \end{pmatrix}$ the transformation equations are

$$\mathbf{x}' = \mathbf{x} + \mathrm{r}$$

$$\mathbf{y}' = \mathbf{y} + \mathrm{s}$$

i.e. a translation is <u>not</u> a linear transformation.

Note

If asked to find the transformation represented by a matrix, find the image of the unit square under this matrix.

Example

Give the geometrical description of the effect of the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Solution

Note

To define a shear you must state

- Invariant line and
- Image of some point not on the invariant line

Note

A shear has an invariant line and all points not on the line move parallel to the line.

- The distance gone is proportional to their distance from the line
- Points on the opposite side of the line go in the opposite direction.

Example

Find the matrices corresponding to each of the following linear transformations:-

(a) Rotation of 90^0 AC about the origin.
(b) Reflection in x-axis.

Solution

Example

A linear transformation T has matrix $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Find

(a) The image of the point (2,3) under T.
(b) The co-ordinate of the point having an image of (7,2) under T.

Solution

Example

Find the 2X2 matrix which will transform (1,2) to (3,3) and (-1,1) to (-3,3).

Solution

Transformation Matrices

1. If a matrix T transforms some shape ABC to its image A'B'C', then the inverse matrix T^{-1} maps A'B'C' to ABC.
2. If a matrix T corresponds to a certain linear transformation, then det. T gives the scale factor for any change of area under the transformation

   i.e. if T transforms ABC to A'B'C' and det T=t, then $Area\ \text{A'B'C'} = \text{Area ABC} \times |t|$

   Thus any matrix with a determinant of 1 leaves the area of a shape unchanged.

   If a matrix is singular i.e. det=0, the shape reduces to a straight line.

Example

Consider the transformation matrix $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and show all the images lie on a straight line.

Solution

3. If a matrix P transforms (x,y) to (x',y') and matrix Q transforms (x',y') to (x'',y'') then the single matrix equivalent to P followed by Q which will take (x,y) to (x'',y'') directly , is given by QP.

   Proof

4. If QP is the matrix representing the combined transformation $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x'' \\ y'' \end{pmatrix}$ then the inverse of

   $QP = (QP)^{-1} = P^{-1}Q^{-1}$

Proof

*UPM p168 Ex6B Q1,3,5-10,13

Further considerations

Invariant points

If a transformation maps some point $A(x, y)$ onto itself, then A is said to be an *invariant point* of the transformation.

Example 6

Find any invariant points of the transformations given by

(a) $\begin{aligned} x' &= 2y - 3 \\ y' &= x + 1 \end{aligned}$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

Solution

Transformation of a line

A linear transformation will map the straight line with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ onto the straight line with vector equation $\mathbf{r} = \mathbf{a}' + \lambda\mathbf{b}'$, i.e. any point lying on $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ will be transformed to a point on the image line $\mathbf{r} = \mathbf{a}' + \lambda\mathbf{b}'$.

All parallel lines will have image lines that are parallel.

Any line that maps onto itself is said to be an **invariant line** of the transformation. It is important to realise that any line which is invariant under a certain transformation need not necessarily be made up of points that are invariant under the transformation. For example, under a stretch parallel to the x-axis, the x-axis itself is an invariant line (as indeed is any line of the form $y = c$). However on the x-axis, only the point (0, 0) is an invariant point under this transformation.

Example

Find the equations of any lines that pass through the origin and map onto themselves under the transformation whose matrix is $\begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution

Example

All points on the line $y = 2x - 3$ are transformed by the matrix $\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$. Find the equation of the image line.

Solutions

Example

All points on the line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ are transformed by the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. Find the equation of the image line.

Solution

Example

(i) Describe the transformation given by the matrix $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

(ii) The curve $5x^2 + y^2 - 4xy - 12x + 6y = 0$ is transformed by the matrix. Show that the image of equation of the image curve is given by $X^2 + Y^2 - 6X = 0$

Solution

Example

(i) The set of points which form a curve whose equation is $x^2 + y^2 - 8x + 8y + 2xy = 0$ is mapped by the matrix $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.

Show that the curve formed by the image points has equation $Y^2 + 8X = 0$.

Solution

*UPM. Page 176 Ex6C Q1,3,4,5,7,8,10,13,14,17

General Reflections and Rotations

1. Rotation about (0,0) through θ A.C,

2 Reflection in a line through the origin

Note

- Any rotation of the x-y plane about the origin can be represented by a matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where $a^2 + b^2 = 1$.

- Any reflection of the x-y plane in a straight line passing through the origin can be represented by a matrix of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where $a^2 + b^2 = 1$.

Example

Find the matrix representing a rotation of 90^0 A.C. about the point A(3,5).

Solution

Example

Find the matrix equation representing an enlargement sf=3 centre (-1,4).

Solution

Example

Find the matrix equation representing a reflection in the line $y = x + 4$.

Solution

Inverse of A 3X3 Matrix

Step1 Find the determinant

- **You find the determinant of a 3 × 3 matrix by reducing the 3 × 3 determinant to 2 × 2 determinants using the formula**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Notice that, in the middle expression in this formula, there is a negative sign before the b.

The 2 × 2 determinant associated with the element a in the first row of the determinant is found by crossing out the row and column in which a lies.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} e & f \\ h & i \end{vmatrix}$$

The 2 × 2 determinant associated with the element b in the first row of the determinant is found by crossing out the row and column in which b lies.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

The 2 × 2 determinant associated with the element c in the first row of the determinant is found by crossing out the row and column in which c lies.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

- **As with 2 × 2 matrices, with 3 × 3 matrices,**
  if det (A) = 0, then A is a singular matrix,
  if det (A) ≠ 0, then A is a non-singular matrix,

Example

Find the determinant of $\begin{pmatrix} 1 & 2 & 7 \\ 3 & -5 & 2 \\ 1 & 1 & 4 \end{pmatrix}$.

Solution

Step2

Form the matrix of the minors of **A**. In this chapter, the symbol **M** is used for the matrix of the minors unless this causes confusion with another matrix in the question.

In forming the matrix of minors **M**, each of the nine elements of the matrix **A** is replaced by its minor.

- The **minor** of an element of a 3×3 matrix is the determinant of the elements which remain when the row and the column containing the element are crossed out.

Step3

From the matrix of minors, form the matrix of **cofactors** by changing the signs of some elements of the matrix of minors according to the **rule of alternating signs** illustrated by the pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

Step4

Write down the transpose of the matrix of cofactors.

Step5

Divide by the determinant.

Example

Find the inverse of $A = \begin{pmatrix} 1 & 2 & 7 \\ 3 & -5 & 2 \\ 1 & 1 & 4 \end{pmatrix}$.

Solution

*P4 Book page 142 Ex4B Q3, 4 (alternate parts)

Transformations in 3-Dimensions

In 3D

$$\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

If the transformation matrix is $\begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix}$

then $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}a\\d\\g\end{pmatrix}$ i.e. the 1st column of the matrix.

$\begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}b\\e\\h\end{pmatrix}$ i.e. the 2nd column of the matrix.

$\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}c\\f\\i\end{pmatrix}$ i.e. the 3rd column of the matrix.

Example

Find the matrix representing the linear transformation

T:$\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x\\-y\end{pmatrix}$

Solution

Example

Find the matrix representing the linear transformation

T:$\begin{pmatrix}x\\y\\z\end{pmatrix} \to 6\begin{pmatrix}x\\x-y\\x+z\end{pmatrix}$

Solution

*P4 Book page 154 Ex4C Q1,4,5,8

2×2 *Linear Equations*

If you have 2x2 equations:-

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Then a unique solution for x and y may always be found using the inverse matrix, where this exists ie. When $\Delta \neq 0$.

Singular case

Ie. When M^{-1} does not exist - $|M|$ is zero

There are 2 case:

1.) $3x - 2y = 1 \ldots.(1)$

$6x - 4y = -1 \ldots.(2)$

$\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Delta = -12 + 12 = 0$

So no inverse , so no unique solution

Reason:- lines are parallel & hence fail to intersect (gradient of each line is $\frac{3}{2}$) i.e. no solution.

2.) $3x - 2y = 1 \ldots.(1)$

$6x - 4y = 2 \ldots.(2)$

$\Delta = -12 + 12 = 0$

So no inverse, so no unique solution

Reason:- line (2) is line (1)x2 therefore lines are coincident and therefore have an infinity of points in common ie. infinite number of solutions $(t, \frac{3t-1}{2})$ where t is a parameter.

Special case

NB. In the case of homogeneous equations

$a_1x + b_1y = 0$

$a_2x + b_2y = 0$

$\Rightarrow \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

these equations possess non-trivial solutions i.e. solutions other than $x = 0, y = 0$ when M^{-1} does not exist ie. $\Delta = 0$.

Result:- The existence of non-trivial solutions $\Leftrightarrow \Delta = 0$

System of $Linear\ Equations$:- 3x3

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

These equations can be represented by :-

$$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \text{ where } M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

And equations have unique solutions provided M^{-1} exists ie. $|M| \neq 0.$

These 3 equations represent 3 planes and if the 3 planes meet in a single point, then the co-ordinates of this point represent a unique solution.

If $|M|$=0 then there is no unique solution and one of the following happens

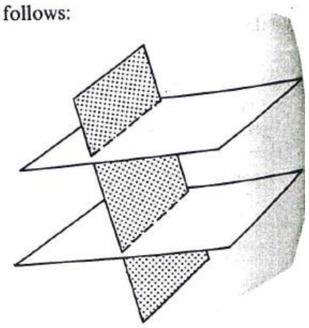
Case 1 :- No solutions-planes have no points in common. 3 ways this can happen.

(i) 3 planes parallel
(ii) 2 planes parallel and 3rd plane intersect them
(iii) 3 planes form a prism (1 plane is parallel to the line of intersection of the other 2)

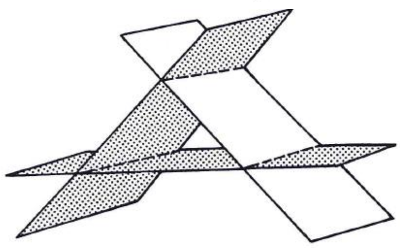
Case 2 :- An infinite number of solutions- 3 planes have a common line(spine of a book), all 3 equations represent the same plane or 2 of the planes are the same with the other crossing (ie. a common line again)

The geometrical interpretation of these two possibilities is as follows:

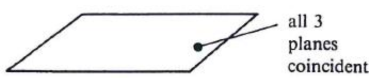
(a) *No solutions* (i.e. inconsistent equations).
This situation will arise when
1. two or more of the planes are parallel (but not coincident). In such cases there is no point common to all three planes.



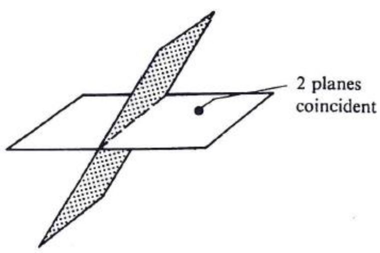
or 2. each plane is parallel to the line of intersection of the other two. Again there will be no point that is common to all three planes.



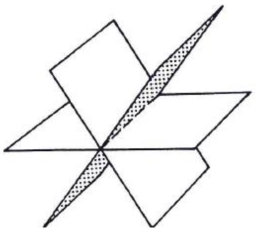
(b) *An infinite number of solutions.*
This situation will arise when
1. all three equations represent the same plane. (i.e. all 3 planes are coincident). Any point in the plane will provide a solution to the equation.



or 2. two of the three planes are coincident and the third plane is not parallel to these two. The planes will intersect in a line and any point on the line will provide a solution.



or 3. the three planes have a common line and any point on this line provides a solution to the equation.



Example

Solve the equations

$2x - 3y + 4z = 1$

$3x - y = 2$

$x + 2y - 4z = 1$

Solution

Example

Solve the equations

$x - 3y - 2z = 9$

$x + 11y + 5z = -5$

$2x + 8y + 3z = 4$

Solution

Homogeneous Equations

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

If $M \neq 0$ then there exists M^{-1}

$M^{-1}M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$I\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ therefor if M^{-1} exists i.e. $|M| \neq 0$ then the solutions are trivial.

So in the homogeneous case Non-trivial solution only exist if $|M| = 0$.

Example

Find the value of *k* for which the following equations have a non-trivial solution and find the solution in that case.

$x + 2y = 0$

$3x + ky - z = 0$

$2x + 5y - 2z = 0$

Solution

*System of Linear Equations question sheet