Further Maths AS1

Matrices

Exercise 6A Questions 4,6-10

4. Find the values of x and y in each of the following matrix equations.

(a) $\begin{pmatrix}3 & -5\\2 & x\end{pmatrix}+\begin{pmatrix}1 & y\\3 & 2\end{pmatrix}=\begin{pmatrix}4 & 6\\5 & -2\end{pmatrix}$ (b) $\begin{pmatrix}2\\y\end{pmatrix}-\begin{pmatrix}x\\3\end{pmatrix}=\begin{pmatrix}4\\-6\end{pmatrix}$

(c) $2\begin{pmatrix}3\\y\end{pmatrix}+\begin{pmatrix}x\\-1\end{pmatrix}=\begin{pmatrix}8\\7\end{pmatrix}$ (d) $\begin{pmatrix}3\\-1\end{pmatrix}+x\begin{pmatrix}-2\\y\end{pmatrix}=\begin{pmatrix}-5\\11\end{pmatrix}$

(e) $\begin{pmatrix}3 & x\\5 & 4\end{pmatrix}\begin{pmatrix}1 & y\\2 & 3\end{pmatrix}=\begin{pmatrix}7 & 3\\13 & 7\end{pmatrix}$ (f) $\begin{pmatrix}2 & 1\\x & 3\end{pmatrix}\begin{pmatrix}3 & 2\\-4 & 3\end{pmatrix}=\begin{pmatrix}2 & 7\\3 & y\end{pmatrix}$

(g) $\begin{pmatrix}2 & x\\3 & -1\end{pmatrix}\begin{pmatrix}2 & 1\\y & -3\end{pmatrix}=\begin{pmatrix}0 & 5\\2 & 6\end{pmatrix}$ (h) $\begin{pmatrix}3 & 2\\4 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}$

(i) $\begin{pmatrix}-1 & 3\\-2 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}9\\10\end{pmatrix}$ (j) $\begin{pmatrix}2 & 4\\-1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\6\end{pmatrix}$

5. By letting $\mathbf{A}=\begin{pmatrix}a & b\\c & d\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}w & x\\y & z\end{pmatrix}$ prove that $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ and that $(\mathbf{AB})^{-1}=\mathbf{B}^{-1}\mathbf{A}^{-1}$.

6. If $\mathbf{A}=\begin{pmatrix}3 & 1\\5 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}6 & 1\\11 & 3\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}-4 & 3\\-5 & 2\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}4 & 7\\-2 & 7\end{pmatrix}$ find the 2×2 matrices **X**, **Y** and **Z** given that $\mathbf{AX}=\mathbf{B}$, $\mathbf{BY}=\mathbf{C}$ and $\mathbf{CZ}=\mathbf{D}$.

7. If $\mathbf{A}=\begin{pmatrix}3 & 2\\-4 & 1\end{pmatrix}$ find the values of m and n given that $\mathbf{A}^2=m\mathbf{A}+n\mathbf{I}$

where **I** is the identity matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

8. Find the possible values x can take given that $\mathbf{A}=\begin{pmatrix}x^2 & 3\\1 & 3x\end{pmatrix}$,

$\mathbf{B}=\begin{pmatrix}3 & 6\\2 & x\end{pmatrix}$ and $\mathbf{AB}=\mathbf{BA}$.

9. Solve the following simultaneous equations by matrix methods.

(a) $x-y=5$, $3x+2y=5$ (b) $x-3y=3$, $5x-9y=11$ (c) $x+3y=1$, $2x-4y=1$

10. If $\mathbf{A}=\begin{pmatrix}-3 & 2 & -1\\2 & -1 & 3\\-1 & 1 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-4 & -3 & 5\\-5 & -4 & 7\\1 & 1 & -1\end{pmatrix}$ find **AB**.

Hence find the values of x, y and z satisfying the three equations

$$-4x-3y+5z=3$$
$$-5x-4y+7z=4$$
$$x+y-z=0.$$

Answers

4. (a) $-4, 11$ (b) $-2, -3$ (c) $2, 4$ (d) $4, 3$ (e) $2, -1$ (f) $5, 19$ (g) $-1, 4$ (h) $2, -1$ (i) $-3, 2$ (j) $-3, 1\frac{1}{2}$

6. $\mathbf{X}=\begin{pmatrix}1 & -1\\3 & 4\end{pmatrix}$ $\mathbf{Y}=\begin{pmatrix}-1 & 1\\2 & -3\end{pmatrix}$ $\mathbf{Z}=\begin{pmatrix}2 & -1\\4 & 1\end{pmatrix}$ **7.** $4, -11$ **8.** $-\frac{1}{2}$ or 3

9. (a) $x=3, y=-2$ (b) $x=1, y=-\frac{2}{3}$ (c) $x=1\frac{1}{2}, y=\frac{1}{2}$ **10.** $\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$, $-1, 2, 1$.

Exercise 6B Questions 1,3,5-10,13

1. The points A(3, 2), B(−1, 4), C(2, 5) and D(1, −1) are transformed to A′, B′, C′, and D′ by the transformation matrix $\begin{pmatrix} 3 & 0 \\ 1 & -3 \end{pmatrix}$. Find the coordinates of A′, B′, C′, and D′.

2. Under a certain transformation the image (x', y') of a point (x, y) is obtained by the rule:

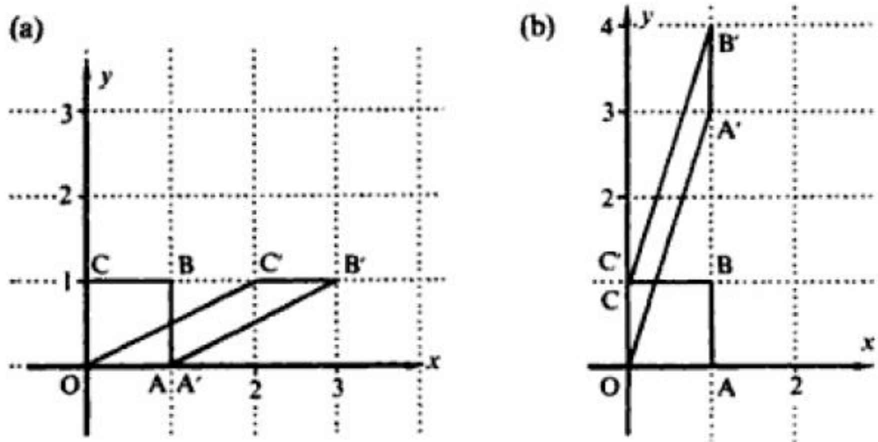
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Find (a) the image of the point (2, 1), (b) the image of the point (−1, 3), (c) the point with an image of (1, −6), (d) the point with an image of (−1, 11).

3. Find the 2 × 2 transformation matrix that will map the point (−2, 3) onto (−7, 6) and (1, −1) onto (3, −1).
Find the image of the point (−1, 3) under this linear transformation and the coordinates of the point that has an image of (6, −2).

5. Find the 2 × 2 transformation matrices **P**, **Q**, **R** and **S** given that
(a) matrix **P** represents a reflection in the *y*-axis,
(b) matrix **Q** represents a 90° clockwise rotation about the origin,
(c) matrix **R** represents a reflection in the line $y = x$,
(d) matrix **S** transforms the point (2, −3) to (4, 14) and the point (1, 3) to (11, −2).
Use your answers to (a), (b) and (c) to show that a reflection in the *y*-axis followed by a 90° clockwise rotation about the origin is equivalent to a reflection in the line $y = x$.

6. The linear transformations shown below are shears transforming OABC to OA′B′C′. For each transformation, write down
(i) the associated 2 × 2 matrix,
(ii) the equations of the transformation in the form $x' = ax + by$
$y' = cx + dy$.



7. A 1st shape is transformed to a 2nd shape by the transformation matrix $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ and a 3rd shape is obtained from the 2nd shape using the transformation matrix $\begin{pmatrix} -1 & 5 \\ -1 & 2 \end{pmatrix}$. Find the single matrix that would transform the 1st shape to the 3rd shape direct. If the 1st shape has an area of 5 sq. units, find the areas of the 2nd and 3rd shapes.

8. Find the matrices corresponding to the following linear transformations.
(a) 180° rotation about the origin,
(b) enlargement scale factor 3, centre (0, 0),
(c) reflection in the line $y = -x$,
(d) stretch (× 2) parallel to the *y*-axis, *x*-axis fixed,
(e) shear with *x*-axis fixed and (0, 1) → (1, 1),
(f) shear with $y = x$ fixed and (1, 0) → (0, −1),
(g) stretch (× 3) perpendicular to $y = -x$ and with $y = -x$ fixed,
(h) shear with $y = x$ fixed and (0, 2) → (4, 6),
(i) shear with $y = 2x$ fixed and (0, 4) → (−2, 0).

9. Give a geometrical description of the effect of each of the following transformation matrices.

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ (i) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ (j) $\begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$

10. Prove that the transformation matrix $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$ maps all points of the x–y plane onto a straight line and find the equation of that line.

13. A certain transformation maps a point A(x, y) onto its image A′(x', y') according to the rule:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

Find (a) the image of the point (2, −1),
(b) the point with an image of (−7, 6),
(c) the coordinates of the point that is mapped onto itself by the transformation.

Answers

1. (9, −3), (−3, −13), (6, −13), (3, 4) 2. (a) (4, 1) (b) (5, 10) (c) (3, −1) (d) (−5, 2)

3. $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, (−5, 9), (2, −2)

4. (a) (5, −3), (15, −9), (13, −7), (3, −1) (b) 8 sq. units (c) $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$

5. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 5 & 2 \\ 4 & -2 \end{pmatrix}$

6. (a) (i) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (ii) $x' = x + 2y$, $y' = y$ (b) (i) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ (ii) $x' = x$, $y' = 3x + y$

7. $\begin{pmatrix} 18 & 14 \\ 6 & 5 \end{pmatrix}$, 10 sq. units, 30 sq. units

8. (a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ (g) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ (i) $\begin{pmatrix} 2 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$

9. (a) reflection in $y = x$
(b) enlargement (× 5) centre (0, 0)
(c) stretch (× 3) parallel to x-axis, y-axis fixed
(d) mapping onto (0, 0)
(e) mapping onto x-axis
(f) mapping onto $y = x$
(g) enlargement (× 3) centre (0, 0) and reflection in $y = x$
(h) shear with y-axis fixed and (1, 0) → (1, 2)
(i) mapping onto $y = 4x$
(j) shear with $y = x$ fixed and (1, 0) → (−2, −3)

10. $2x + 3y = 0$ 12. (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (b) $x' = -x + 2$, $y' = y + 3$

13. (a) (−1, −1) (b) (−3, 2) (c) (−1, 2)

Exercise 6C **Questions 1,3,4,5,6,8,10,13,14,17**

1. Find any invariant points of the transformations given by

(a) $x' = 3y + 2$
$y' = 2x - y + 4$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

2. Two transformations **P** and **T** transform the point (x, y) to its image (x', y') according to the following rules:

$\mathbf{P}\begin{cases} x' = 3x + y \\ y' = x - 1 \end{cases}$ $\mathbf{T}\begin{cases} x' = 2x - y - 3 \\ y' = 1 - x \end{cases}$

Express the transformations **P**, **T**, **PT** and **TP** in the form

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} * \\ * \end{pmatrix}$

Find any invariant points under the transformation
(a) **P**, (b) **T**, (c) **PT**, (d) **TP**.

3. Find the equations of the image lines formed when the lines $y = 2x + 1$ and $3y = 2x - 1$ are transformed using the matrix $\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$.

4. Find the equation of the image line produced by translating all of the points on the line $y = 3x - 1$ by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

5. Find vector equations of the image lines formed when the lines $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are transformed using the matrix $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$.

6. Find the 2×2 transformation matrix **T** that maps $(3, -1)$ onto $(13, -7)$ and $(-1, 3)$ onto $(1, 5)$. Find the equations of the lines obtained when **T** is applied to $y = x$, $y + 2x = 3$ and $y = 2x + 3$.

7. The transformation matrix $\begin{pmatrix} 3 & -2 \\ 3 & -1 \end{pmatrix}$ transforms a line **L** to the line $y = 2x + 3$. Find the equation of **L**.

8. Find the vector equation of the line which, when transformed by the matrix $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, has an image line $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

9. A transformation **T** assigns to any point (x, y) an image (x', y') according to the rule $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix}$.

(a) Find the equation of the image lines obtained when all points on the lines $y = 2x$ and $y = x - 3$ undergo the transformation **T**.

(b) Prove that the line $y = 3x + 1$ maps onto itself under the transformation **T**.

10. Prove that the transformation matrix $\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$ maps all points on the line $y = 3x$ onto themselves.

11. Prove that all lines of the form $y = x + a$ are mapped onto themselves under the transformation given by the matrix $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$.

12. Prove that the lines $y = x + 5$ and $y + 3x = 1$ are mapped onto themselves under the transformation that maps (x, y) onto (x', y') according to the relationship $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -9 \end{pmatrix}$.

13. Show that the transformation with matrix $\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}$ maps all points on the line $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ onto a single point and find the position vector of this point.

14. Show that the transformation with matrix $\begin{pmatrix} -4 & 2 \\ 6 & -3 \end{pmatrix}$ maps all points on the line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j})$ onto a single point and find the position vector of this point.

15. Prove that all points on the line $y + 3x + 2 = 0$ are mapped onto a single point under the transformation that maps (x, y) onto (x', y') according to the relationship $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and find the coordinates of this single point.

16. Find the equations of any straight lines that pass through the origin and that map onto themselves under the transformation defined by the matrix

(a) $\begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 3 \\ 1 & -5 \end{pmatrix}$ (c) $\begin{pmatrix} a & 1 \\ 8 & a \end{pmatrix}$

17. Find the 2×2 matrix that maps $(1, 2)$ onto $(-3, 0)$ and $(-2, -3)$ onto $(2, -1)$. For the transformation defined by this matrix, find the equations of two invariant straight lines passing through the origin. Which of these two lines is a set of invariant points under the transformation?

Answers

1. (a) $(-4, -2)$ (b) $(-3, 2)$ (c) $(4\frac{1}{2}, 1)$

2. $\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -8 \\ -4 \end{pmatrix}$,

$\begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(a) $(\frac{1}{3}, -\frac{2}{3})$ (b) $(2, -1)$ (c) $(2, 0)$ (d) $(1, -1)$

3. $7y = 5x + 6$, $x = 3y + 2$ **4.** $y = 3x - 4$ **5.** $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 11 \\ 1 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 5 \end{pmatrix}$

6. $\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$, $x + 7y = 0$, $4x + y = 27$, $y = 3$ **7.** $y = x + 1$ **8.** $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

9. (a) $3y = 10x + 8$, $y = 5x + 21$ **13.** $\begin{pmatrix} 7 \\ -7 \end{pmatrix}$ **14.** $-2\mathbf{i} + 3\mathbf{j}$ **15.** $(-1, 1)$

16. (a) $y = x$, $y = 2x$ (b) $3y = x$, $y + x = 0$ (c) $y = \pm 2\sqrt{2}x$

17. $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$, $y = x$, $2y = x$; $y = x$ **18.** 8, $y = 2x$

19. (a) $y = x + 6$, $3y = 2x + 12$ (b) $y + 4x = 1$, $y = x + 1$ (c) $y = 2$, $y = 2x - 6$
(d) $y = x - 1$ (e) $y = 3x - 9$

Exercise 6D **Questions 1a,2a,3,4**

1. Find the matrices which represent the following linear transformations.
   (a) a rotation of 30° anticlockwise about the origin,
   (b) a rotation of 45° anticlockwise about the origin,
   (c) a rotation of 120° anticlockwise about the origin.
2. Find the matrices which represent the following linear transformations,
   (a) a reflection in the line $y = \sqrt{3}x$,
   (b) a reflection in the line $\sqrt{3}y = x$,
   (c) a reflection in the line $y = 2x$.
3. Give a geometrical description of the transformations corresponding to the following matrices

   (a) $\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$

4. Find the matrix equations representing each of the following transformations,
   (a) a rotation of 90° anticlockwise about the point $(-1, 4)$,
   (b) a rotation of 180° about the point $(3, -1)$,
   (c) an enlargement, scale factor 3, centre $(2, -1)$,
   (d) a reflection in the line $y = x + 5$,
   (e) a glide reflection in the line $y = x + 5$ with the point $(0, 5)$ mapped onto $(3, 8)$.

Answers

1. (a) $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (c) $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

2. (a) $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ (c) $\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

3. (a) rotation of 53·1° anticlockwise about origin (b) reflection in $2y = x$
   (c) rotation of 53·1° anticlockwise about origin and enlargement (× 5), centre the origin

4. (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

   (c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ (d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 5 \end{pmatrix}$

   (e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 8 \end{pmatrix}$

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3 Evaluate:

(a) $\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$ (d) $\begin{vmatrix} 0 & -2 \\ 1 & -4 \end{vmatrix}$

(e) $\begin{vmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ -1 & 4 & -6 \end{vmatrix}$ (f) $\begin{vmatrix} -2 & 7 & 3 \\ 1 & 2 & 4 \\ -1 & 2 & 0 \end{vmatrix}$

(g) $\begin{vmatrix} 2 & -3 & 6 \\ -2 & 4 & 5 \\ -1 & 0 & -5 \end{vmatrix}$ (h) $\begin{vmatrix} 1 & 2 & -3 \\ 2 & 2 & -4 \\ -4 & 2 & 1 \end{vmatrix}$

(i) $\begin{vmatrix} -3 & 1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -2 \end{vmatrix}$ (j) $\begin{vmatrix} 3 & 4 & 1 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{vmatrix}$

(k) $\begin{vmatrix} 1 & -1 & 3 \\ 2 & -2 & 4 \\ 3 & -3 & 5 \end{vmatrix}$ (l) $\begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 7 \\ -2 & -7 & 0 \end{vmatrix}$

(m) $\begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{vmatrix}$ (n) $\begin{vmatrix} 3 & 0 & 2 \\ 0 & -1 & 4 \\ 1 & 1 & -2 \end{vmatrix}$

(o) $\begin{vmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix}$ (p) $\begin{vmatrix} 5 & 1 & 3 \\ -2 & 0 & 1 \\ 1 & 1 & -2 \end{vmatrix}$

4 Find the inverse of:

(a) $\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 2 \\ -1 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -3 \\ 1 & -4 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} -3 & 7 \\ 9 & 22 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & -2 \\ 1 & -5 & 4 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & 2 & 3 \\ 1 & 1 & 2 \\ 5 & -1 & 4 \end{pmatrix}$

(i) $\begin{pmatrix} -2 & 3 & -4 \\ 1 & 2 & -3 \\ -3 & 0 & -2 \end{pmatrix}$ (j) $\begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix}$

(k) $\begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$ (l) $\begin{pmatrix} 4 & -5 & 2 \\ 0 & 1 & -7 \\ 1 & 1 & -2 \end{pmatrix}$

(m) $\begin{pmatrix} 3 & 2 & -3 \\ 1 & 1 & -4 \\ 2 & 2 & -6 \end{pmatrix}$ (n) $\begin{pmatrix} -2 & 1 & -2 \\ 4 & 3 & 1 \\ 0 & 1 & -6 \end{pmatrix}$

(o) $\begin{pmatrix} 2 & 1 & -3 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ (p) $\begin{pmatrix} 2 & 2 & 1 \\ 4 & 1 & 5 \\ -1 & 1 & 7 \end{pmatrix}$

Answers

3 (a) 1 (b) −2 (c) −5 (d) 2
(e) −9 (f) 0 (g) 29 (h) 2
(i) −36 (j) −29 (k) 0 (l) 0
(m) 0 (n) −4 (o) 6 (p) −14

4 (a) $\frac{1}{13}\begin{pmatrix} 4 & -5 \\ 1 & 2 \end{pmatrix}$

(b) $-\frac{1}{19}\begin{pmatrix} 7 & -2 \\ 1 & -3 \end{pmatrix}$

(c) $-\frac{1}{5}\begin{pmatrix} -4 & 3 \\ -1 & 2 \end{pmatrix}$

(d) $\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$

(e) $-\frac{1}{129}\begin{pmatrix} 22 & -7 \\ -9 & -3 \end{pmatrix}$

(f) $-\frac{1}{3}\begin{pmatrix} 0 & -3 & -2 \\ -3 & 6 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

(g) $\frac{1}{6}\begin{pmatrix} -10 & -13 & -4 \\ -2 & -5 & -2 \\ 0 & -3 & 0 \end{pmatrix}$

(h) $-\frac{1}{12}\begin{pmatrix} 6 & -11 & 1 \\ 6 & -19 & 5 \\ -6 & 9 & -3 \end{pmatrix}$

(i) $\frac{1}{17}\begin{pmatrix} -4 & 6 & -1 \\ 11 & -8 & -10 \\ 6 & -9 & -7 \end{pmatrix}$

(j) $\frac{1}{5}\begin{pmatrix} 2 & -1 & 1 \\ 5 & -10 & 0 \\ 3 & -4 & -1 \end{pmatrix}$

(k) $\frac{1}{3}\begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

(l) $\frac{1}{53}\begin{pmatrix} 5 & -8 & 33 \\ -7 & -10 & 28 \\ -1 & -9 & 4 \end{pmatrix}$

P4 Book Ex4C Questions 1,4,5,8

1 Given that

$$T:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x+y\\x-y\end{pmatrix}$$

$$U:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}x\\2x+y\end{pmatrix}$$

find the matrix that represents (a) T (b) U (c) TU (d) UT.

4 Given that

$$T:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}y\\x\end{pmatrix}$$

$$U:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}2x\\y\end{pmatrix}$$

find the matrix that represents (a) T^{-1} (b) U^{-1} (c) $(UT)^{-1}$.

5 Given that

$$T:\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}2x+z\\y\\-y+z\end{pmatrix}$$

$$U:\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}-x+2y-3z\\2x-y+4z\\3x+4y+z\end{pmatrix}$$

find the matrix that represents (a) T (b) U (c) TU (d) UT.

8 Given that

$$T:\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x-y+3z\\2x+y+4z\\y+z\end{pmatrix}$$

$$U:\begin{pmatrix}x\\y\\z\end{pmatrix}\mapsto\begin{pmatrix}x+3y-2z\\-2x-9y+5z\\x+10y+4z\end{pmatrix}$$

find the matrix that represents

(a) T^{-1} (b) U^{-1} (c) $(TU)^{-1}$ (d) $(UT)^{-1}$.

9 Given that

$$\mathbf{A}=\begin{pmatrix}1&0&2\\4&1&7\\-2&3&6\end{pmatrix}\quad\text{and}\quad\mathbf{B}=\begin{pmatrix}-2&1&3\\4&-9&5\\1&1&-2\end{pmatrix}$$

find (a) $\mathbf{A}^{\mathrm{T}}$ (b) $\mathbf{B}^{\mathrm{T}}$.

Hence find (c) $(\mathbf{AB})^{\mathrm{T}}$ (d) $(\mathbf{BA})^{\mathrm{T}}$.

10 Given that

$$\mathbf{A}=\begin{pmatrix}2&-1&3\\1&4&-7\\6&6&-3\end{pmatrix}\quad\text{and}\quad\mathbf{B}=\begin{pmatrix}1&2&4\\3&1&-2\\-1&1&3\end{pmatrix}$$

find (a) $\mathbf{A}^{\mathrm{T}}$ (b) $\mathbf{B}^{\mathrm{T}}$.

Hence find (c) $(\mathbf{AB})^{\mathrm{T}}$ (d) $(\mathbf{BA})^{\mathrm{T}}$.

Answers

1 (a) $\begin{pmatrix}1&1\\1&-1\end{pmatrix}$ (b) $\begin{pmatrix}1&0\\2&1\end{pmatrix}$

(c) $\begin{pmatrix}3&1\\-1&-1\end{pmatrix}$ (d) $\begin{pmatrix}1&1\\3&1\end{pmatrix}$

4 (a) $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ (b) $\begin{pmatrix}\frac{1}{2}&0\\0&1\end{pmatrix}$

(c) $\begin{pmatrix}0&1\\\frac{1}{2}&0\end{pmatrix}$

5 (a) $\begin{pmatrix}2&0&1\\0&1&0\\0&-1&1\end{pmatrix}$

(b) $\begin{pmatrix}-1&2&-3\\2&-1&4\\3&4&1\end{pmatrix}$

(c) $\begin{pmatrix}1&8&-5\\2&-1&4\\1&5&-3\end{pmatrix}$

(d) $\begin{pmatrix}-2&5&-4\\4&-5&6\\6&3&4\end{pmatrix}$

6 (a) $\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}\\0&1&0\\0&1&1\end{pmatrix}$

(b) $\frac{1}{4}\begin{pmatrix}-17&-14&5\\10&8&-2\\11&10&-3\end{pmatrix}$

(c) $\frac{1}{8}\begin{pmatrix}-17&-1&27\\10&2&-14\\11&3&-17\end{pmatrix}$

(d) $\frac{1}{8}\begin{pmatrix}-38&-32&10\\20&16&-4\\42&36&-10\end{pmatrix}$

7 (a) $\begin{pmatrix}1&-1&3\\2&1&4\\0&1&1\end{pmatrix}$

(b) $\begin{pmatrix}1&3&-2\\-2&-9&5\\1&10&4\end{pmatrix}$

(c) $\begin{pmatrix}6&42&5\\4&37&17\\-1&1&9\end{pmatrix}$

(d) $\begin{pmatrix}7&0&13\\-20&-2&-37\\21&13&47\end{pmatrix}$

8 (a) $\frac{1}{5}\begin{pmatrix}-3&4&-7\\-2&1&2\\2&-1&3\end{pmatrix}$

(b) $\frac{1}{25}\begin{pmatrix}86&32&3\\-13&-6&1\\11&7&3\end{pmatrix}$

(c) $-\frac{1}{125}\begin{pmatrix}316&-373&529\\-53&59&-82\\41&-48&54\end{pmatrix}$

(d) $-\frac{1}{125}\begin{pmatrix}387&169&26\\163&56&-1\\-218&-91&-14\end{pmatrix}$

9 (a) $\begin{pmatrix}1&4&-2\\0&1&3\\2&7&6\end{pmatrix}$

(b) $\begin{pmatrix}-2&4&1\\1&-9&1\\3&5&-2\end{pmatrix}$

(c) $\begin{pmatrix}0&3&22\\3&2&-23\\-1&3&-3\end{pmatrix}$

(d) $\begin{pmatrix}-4&-42&9\\10&6&-5\\21&-25&-3\end{pmatrix}$

10 (a) $\begin{pmatrix}2&1&6\\-1&4&6\\3&-7&-3\end{pmatrix}$

(b) $\begin{pmatrix}1&3&-1\\2&1&1\\4&-2&3\end{pmatrix}$

(c) $\begin{pmatrix}-4&20&27\\6&-1&15\\19&-25&3\end{pmatrix}$

(d) $\begin{pmatrix}28&-5&17\\31&-11&23\\-23&8&-19\end{pmatrix}$

System of Linear Equations Questions

Q1

Consider the following system of simultaneous equations

$$x - y + 2z = 6$$
$$2x + 3y - z = 7$$
$$x + 9y - 8z = -4$$

i) By evaluating an appropriate determinant, show that this system does not have a unique solution.

ii) Solve this system of simultaneous equations. (NICCEA)

Q2

Consider the system of simultaneous equations

$$3x + y - 2z = -4$$
$$x + 2y + 3z = 11$$
$$3x - 4y - 13z = -41$$

i) Solve this system of equations.

ii) Hence show in a sketch how the planes defined by the above equations are arranged so that the solution is of the form found in part i. (NICCEA)

Q3

Show that the equations

$$x + \lambda y + z = 2a$$
$$x + y + \lambda z = 2b$$
$$\lambda x + y + \lambda z = 2c$$

where $a, b, c \in \mathbb{R}$, have a unique solution for x, y, z provided that $\lambda \neq 1$ and $\lambda \neq -1$.

a) In the case when $\lambda = 1$, state the condition to be satisfied by a, b and c for the equations to be consistent.

b) In the case when $\lambda = -1$, show that for the equations to be consistent

$$a + c = 0$$

Solve the equations in this case.

Give a geometrical description of the configuration of the three planes represented by the equations in the cases:

i) $\lambda = -1$ and $a + c = 0$

ii) $\lambda = -1$ and $a + c \neq 0$. (NEAB)

Q4

Find the values of k for which the simultaneous equations

$$kx + 2y + z = 0$$
$$3x - 2z = 4$$
$$3x - 6ky - 4z = 14$$

do not have a unique solution for x, y and z.

Show that, when $k = -2$, the equations are inconsistent, and give a geometrical interpretation of the situation in this case. (OCR)

Q5

Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -k \\ 1 & -k & -1 \end{pmatrix}$$

find $\det \mathbf{M}$ in terms of k.

Determine the values of k for which the simultaneous equations

$$x + y - z = 1$$
$$x + 2y - kz = 0$$
$$x - ky - z = 1$$

have a unique solution.

i) Solve these equations in the case when $k = 2$.

ii) Show that the equations have no solution when $k = 1$.

iii) Find the general solution when $k = -1$.

Give a geometrical interpretation of the equations in each of the three cases $k = 2$, $k = 1$ and $k = -1$. (NEAB)

Q6

Show that the only real value of λ for which the simultaneous equations

$$(2 + \lambda)x - y + z = 0$$
$$x - 2\lambda y - z = 0$$
$$4x - y - (\lambda - 1)z = 0$$

have a solution other than $x = y = z = 0$ is -1.

Solve the equations in the case when $\lambda = -1$, and interpret your result geometrically. (NEAB)

Q7

Consider the system of equations x, y and z,

$$2x + 3y - z = p$$
$$x - 2z = -5$$
$$qx + 9y + 5z = 8$$

where p and q are real.

Find the values of p and q for which this system has:

i) a unique solution

ii) an infinite number of solutions

iii) no solution. (NICCEA)

Answers

1(i) det =0

(ii) (4-t,t,t+1) i.e. Solution are on a straight line (spine of a book)

2 (i) $(\frac{7t-19}{5},\frac{37-11t}{5},t)$

(ii) Spine of a book

3(a) a=b=c

(b) (i) Equation 1 and 2 are the same and 3 is a non-parallel plane hence an infinite number of solutions

(ii) 2 parallel planes and the other plane intersecting both

4 Triangular prism

5(i) planes met at a point

(ii)Triangular prism

(iii) 2 parallel planes and the other plane intersecting both

6 Spine of a book

7(i) $q \neq 2$

(ii) q=2, p=-4

(iii) q=2, $p \neq -4$