A-Level Further Maths

AS1 Notes

Topics

Matrices

- A **matrix** can be thought of as an array of numbers (a collection of numbers set out in a table) and they come in different shapes and sizes.
- You can describe these different shapes and sizes in terms of the **dimension** of the matrix. This is given by two numbers n and m in the form $n \times m$ (read as n by m), where n is the number of rows (horizontal or across the page) and m is the number of columns (vertical or down the page) in the matrix.
- An $n \times m$ matrix has n rows and m columns.
- Matrices are usually denoted in bold print with a capital letter e.g. **A**, **M** etc.

Example

Solution

Give the dimensions of the following matrices

a $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, **b** $\begin{pmatrix} 1 & 0 & 2 \end{pmatrix}$,

c $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$, **d** $\begin{pmatrix} 3 & 2 \\ -1 & 1 \\ 0 & -3 \end{pmatrix}$.

You can add and subtract matrices of the same dimension.

Example

Solution

Find **a** $\begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 4 \\ 5 & 3 \end{pmatrix}$

b $\begin{pmatrix} 1 & -3 & 4 \\ 2 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 & 1 \\ 5 & 2 & 3 \end{pmatrix}$

- **To multiply a matrix by a number you simply multiply each element of the matrix by that number.**

Example

Solution

$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & 0 & -4 \end{pmatrix}$

Find **a** $2\mathbf{A}$ **b** $\frac{1}{2}\mathbf{B}$

Multiplying Matrices

- The basic operation consists of multiplying each element in the **row** of the left hand matrix by each corresponding element in the **column** of the right hand matrix and adding the results together.
- The number of columns in the left hand matrix must equal the number of rows in the right hand matrix.
- The product will then have the same number of rows as the left hand matrix and the same number of columns as the right hand matrix.

***Matrices can only be multiplied if the no. of columns of 1st matrix is equal to the no. of rows of the 2nd.**

***Matrices which can be multiplied are said to be conformable.**

So if

$\mathbf{A} \times \mathbf{B} = \mathbf{C}$

Dimensions: $(n \times m) \times (m \times k) \quad (n \times k)$

n is from the number of rows in **A**.
k is from the number of columns in **B**.

These numbers must be the same.

Example

Given that $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 0 & -2 \end{pmatrix}$ find

a **AB**

b **BA.**

Solution

*Matrix multiplication is not Commutative i.e. AB≠BA

The Identity Matrix

The 2X2 Identity Matrix is **I**=$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The 3X3 Identity Matrix is **I**=$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Where AI=IA=A *so multiplying a matrix BY I is a bit like multiplying a number by 1.

Inverse of a 2X2

- If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
  and then $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$

The value of $ad - bc$ is called the **determinant** of **A** and written det(**A**).

- $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(\mathbf{A}) = ad - bc$ so $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Notice that if det(**A**) = 0 you will not be able to find $\mathbf{A}^{-1}$ because $\frac{1}{\det(\mathbf{A})}$ is not defined, in such cases we say **A** is **singular**.

- If det(**A**) = 0, then **A** is a **singular matrix** and $\mathbf{A}^{-1}$ cannot be found.
  If det(**A**) ≠ 0, then **A** is a **non-singular matrix** and $\mathbf{A}^{-1}$exists.

Transpose

- Given a matrix **A**, you form the **transpose** of the matrix $\mathbf{A}^{\mathrm{T}}$, by interchanging the rows and the columns of **A**. You take the first row of **A** and write it as the first column of $\mathbf{A}^{\mathrm{T}}$, you take the second row of **A** and write it as the second column of $\mathbf{A}^{\mathrm{T}}$, and so on.

  If $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 0 & -3 \\ 1 & 5 \end{pmatrix}$, then $\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -3 & 5 \end{pmatrix}$.

- The transpose of a matrix of dimension $n \times m$ is a matrix of dimension $m \times n$.

- The transpose of a square matrix is another square matrix with the same dimensions. For example, the transpose of a 2×2 matrix is another 2×2 matrix.

- If $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$, the matrix **A** is **symmetric**.

Example

Use an inverse matrix to solve the simultaneous equations

$$4x - y = 1$$

$$-2x + 3y = 12$$

Solution

*UPM p.161 Ex6A Q4,6-10

Transformations

A transformation of a plane takes any point A in the plane and maps it onto one and only one image A′ lying in the plane. We say that the point A(x,y) position vector $\begin{pmatrix} x \\ y \end{pmatrix}$ has an image A'(x',y') position vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$ under the transformation.

A transformation is said to be 'linear' if any linear relationship between position vectors is conserved under the transformation i.e. if a linear transformation maps a point A position vector $\boldsymbol{a}$ onto its image A' position vector $\boldsymbol{a'}$ and if $\boldsymbol{a} = \mu\boldsymbol{p} + \alpha\boldsymbol{q}$ then $\boldsymbol{a}' = \mu\boldsymbol{p}' + \alpha\boldsymbol{q'}$ where $\boldsymbol{p'}$ ***and*** $\boldsymbol{q'}$ are the images of $\boldsymbol{p}$ *and* $\boldsymbol{q}$**.**

All linear transformations of the plane can be expressed as a pair of equations of the form:-

$$\boldsymbol{x}' = ax + by$$

$$\boldsymbol{y}' = cx + dy$$

Or writing this in matrix form

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Note

1. The transformation matrix can be found by finding the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ because $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$
   i.e. the image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the first column of the transformation matrix
   and the image of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the second column of the transformation matrix.

2. Under any linear transformation the origin (0,0) maps to itself.
3. Under a translation by the vector $\begin{pmatrix} r \\ s \end{pmatrix}$ the transformation equations are

$$\mathbf{x}' = \mathbf{x} + \mathrm{r}$$

$$\mathbf{y}' = \mathbf{y} + \mathrm{s}$$

i.e. a translation is not a linear transformation.

Note

If asked to find the transformation represented by a matrix, find the image of the unit square under this matrix.

Example

Give the geometrical description of the effect of the matrix $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Solution

Note

To define a shear you must state

- Invariant line and
- Image of some point not on the invariant line

Note

A shear has an invariant line and all points not on the line move parallel to the line.

- The distance gone is proportional to their distance from the line
- Points on the opposite side of the line go in the opposite direction.

Example

Find the matrices corresponding to each of the following linear transformations:-

(a) Rotation of 90^0 AC about the origin.
(b) Reflection in x-axis.

Solution

Example

A linear transformation T has matrix $\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Find

(a) The image of the point (2,3) under T.
(b) The co-ordinate of the point having an image of (7,2) under T.

Solution

Example

Find the 2X2 matrix which will transform (1,2) to (3,3) and (-1,1) to (-3,3).

Solution

Transformation Matrices

1. If a matrix T transforms some shape ABC to its image A'B'C', then the inverse matrix T^{-1} maps A'B'C' to ABC.
2. If a matrix T corresponds to a certain linear transformation, then det. T gives the scale factor for any change of area under the transformation

   i.e. if T transforms ABC to A'B'C' and det T=t, then $Area\ \text{A'B'C'} = \text{Area ABC} \times |t|$

   Thus any matrix with a determinant of 1 leaves the area of a shape unchanged.

   If a matrix is singular i.e. det=0, the shape reduces to a straight line.

Example

Consider the transformation matrix $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ and show all the images lie on a straight line.

Solution

3. If a matrix P transforms (x,y) to (x',y') and matrix Q transforms (x',y') to (x'',y'') then the single matrix equivalent to P followed by Q which will take (x,y) to (x'',y'') directly , is given by QP.

   Proof

4. If QP is the matrix representing the combined transformation $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x'' \\ y'' \end{pmatrix}$ then the inverse of

   $QP = (QP)^{-1} = P^{-1}Q^{-1}$

Proof

*UPM p168 Ex6B Q1,3,5-10,13

Further considerations

Invariant points

If a transformation maps some point A(x, y) onto itself, then A is said to be an *invariant point* of the transformation.

Example 6

Find any invariant points of the transformations given by

(a) $\begin{aligned} x' &= 2y - 3 \\ y' &= x + 1 \end{aligned}$ (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

<u>Solution</u>

Transformation of a line

A linear transformation will map the straight line with vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ onto the straight line with vector equation $\mathbf{r} = \mathbf{a}' + \lambda\mathbf{b}'$, i.e. any point lying on $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ will be transformed to a point on the image line $\mathbf{r} = \mathbf{a}' + \lambda\mathbf{b}'$.

All parallel lines will have image lines that are parallel.

Any line that maps onto itself is said to be an **invariant line** of the transformation. It is important to realise that any line which is invariant under a certain transformation need not necessarily be made up of points that are invariant under the transformation. For example, under a stretch parallel to the x-axis, the x-axis itself is an invariant line (as indeed is any line of the form $y = c$). However on the x-axis, only the point (0, 0) is an invariant point under this transformation.

<u>Example</u>

Find the equations of any lines that pass through the origin and map onto themselves under the transformation whose matrix is $\begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}$.

<u>Solution</u>

Example

All points on the line $y = 2x - 3$ are transformed by the matrix $\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$. Find the equation of the image line.

Solutions

Example

All points on the line $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ are transformed by the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$. Find the equation of the image line.

Solution

Example

(i) Describe the transformation given by the matrix $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

(ii) The curve $5x^2 + y^2 - 4xy - 12x + 6y = 0$ is transformed by the matrix. Show that the image of equation of the image curve is given by $X^2 + Y^2 - 6X = 0$

Solution

Example

(i) The set of points which form a curve whose equation is $x^2 + y^2 - 8x + 8y + 2xy = 0$ is mapped by the matrix $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$.

Show that the curve formed by the image points has equation $Y^2 + 8X = 0$.

Solution

*UPM. Page 176 Ex6C Q1,3,4,5,7,8,10,13,14,17

General Reflections and Rotations

1. Rotation about (0,0) through θ A.C,

2 Reflection in a line through the origin

Note

- Any rotation of the x-y plane about the origin can be represented by a matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where $a^2 + b^2 = 1$.

- Any reflection of the x-y plane in a straight line passing through the origin can be represented by a matrix of the form $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ where $a^2 + b^2 = 1$.

Example
Find the matrix representing a rotation of 90^0 A.C. about the point A(3,5).

Solution

Example

Find the matrix equation representing an enlargement sf=3 centre (-1,4).

Solution

Example

Find the matrix equation representing a reflection in the line $y = x + 4$.

Solution

Inverse of A 3X3 Matrix

Step1 Find the determinant

■ **You find the determinant of a 3 × 3 matrix by reducing the 3 × 3 determinant to 2 × 2 determinants using the formula**

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Notice that, in the middle expression in this formula, there is a negative sign before the b.

The 2 × 2 determinant associated with the element a in the first row of the determinant is found by crossing out the row and column in which a lies.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} e & f \\ h & i \end{vmatrix}$$

The 2 × 2 determinant associated with the element b in the first row of the determinant is found by crossing out the row and column in which b lies.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

The 2 × 2 determinant associated with the element c in the first row of the determinant is found by crossing out the row and column in which c lies.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

■ **As with 2 × 2 matrices, with 3 × 3 matrices,**
if det (A) = 0, then A is a singular matrix,
if det (A) ≠ 0, then A is a non-singular matrix,

Example

Find the determinant of $\begin{pmatrix} 1 & 2 & 7 \\ 3 & -5 & 2 \\ 1 & 1 & 4 \end{pmatrix}$.

Solution

Step2

Form the matrix of the minors of **A**. In this chapter, the symbol **M** is used for the matrix of the minors unless this causes confusion with another matrix in the question.

In forming the matrix of minors **M**, each of the nine elements of the matrix **A** is replaced by its minor.

- The **minor** of an element of a 3 × 3 matrix is the determinant of the elements which remain when the row and the column containing the element are crossed out.

Step3

From the matrix of minors, form the matrix of **cofactors** by changing the signs of some elements of the matrix of minors according to the **rule of alternating signs** illustrated by the pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

Step4

Write down the transpose of the matrix of cofactors.

Step5

Divide by the determinant.

Example

Find the inverse of $A = \begin{pmatrix} 1 & 2 & 7 \\ 3 & -5 & 2 \\ 1 & 1 & 4 \end{pmatrix}$.

Solution

*P4 Book page 142 Ex4B Q3, 4 (alternate parts)

Transformations in 3-Dimensions

In 3D

$$\boldsymbol{i} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{j} = \begin{pmatrix}0\\1\\0\end{pmatrix}, \boldsymbol{k} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

If the transformation matrix is $\begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix}$

then $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}a\\d\\g\end{pmatrix}$ i.e. the 1st column of the matrix.

$\begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}b\\e\\h\end{pmatrix}$ i.e. the 2nd column of the matrix.

$\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}c\\f\\i\end{pmatrix}$ i.e. the 3rd column of the matrix.

Example

Find the matrix representing the linear transformation

T:$\begin{pmatrix}x\\y\end{pmatrix} \to \begin{pmatrix}x\\-y\end{pmatrix}$

Solution

Example

Find the matrix representing the linear transformation

T:$\begin{pmatrix}x\\y\\z\end{pmatrix} \to 6\begin{pmatrix}x\\x-y\\x+z\end{pmatrix}$

Solution

*P4 Book page 154 Ex4C Q1,4,5,8

2×2 *Linear Equations*

If you have 2x2 equations:-

$a_1 x + b_1 y = c_1$

$a_2 x + b_2 y = c_2$

Then a unique solution for x and y may always be found using the inverse matrix, where this exists ie. When $\Delta \neq 0$.

<u>Singular case</u>

Ie. When M^{-1} does not exist - $|M|$ is zero

There are 2 case:

1.) $3x - 2y = 1 \ldots .(1)$

$6x - 4y = -1 \ldots .(2)$

$$\begin{pmatrix} 3 & -2 \\ 6 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\Delta = -12 + 12 = 0$

So no inverse , so no unique solution

Reason:- lines are parallel & hence fail to intersect (gradient of each line is $\frac{3}{2}$) i.e. no solution.

2.) $3x - 2y = 1 \ldots .(1)$

$6x - 4y = 2 \ldots .(2)$

$\Delta = -12 + 12 = 0$

So no inverse, so no unique solution

Reason:- line (2) is line (1)x2 therefore lines are coincident and therefore have an infinity of points in common ie. infinite number of solutions $(t, \frac{3t-1}{2})$ where t is a parameter.

<u>*Special case*</u>

NB. In the case of homogeneous equations

$a_1 x + b_1 y = 0$

$a_2 x + b_2 y = 0$

$$=> \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

these equations possess non-trivial solutions i.e. solutions other than $x = 0, y = 0$ when M^{-1} does not exist ie. $\Delta = 0$.

Result:- The existence of non-trivial solutions $<=> \Delta = 0$

<u>System of $Linear\ Equations$:- 3x3</u>

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

These equations can be represented by :-

$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ where $M = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

And equations have unique solutions provided M^{-1} exists ie. $|M| \neq 0$.

These 3 equations represent 3 planes and if the 3 planes meet in a single point, then the co-ordinates of this point represent a unique solution.

If $|M|$=0 then there is no unique solution and one of the following happens

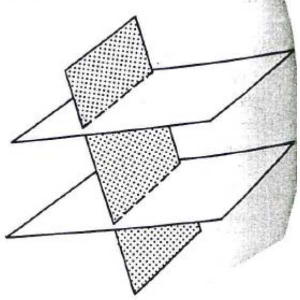
<u>Case 1 :- </u>No solutions-planes have no points in common. 3 ways this can happen.

(i) 3 planes parallel
(ii) 2 planes parallel and 3rd plane intersect them
(iii) 3 planes form a prism (1 plane is parallel to the line of intersection of the other 2)

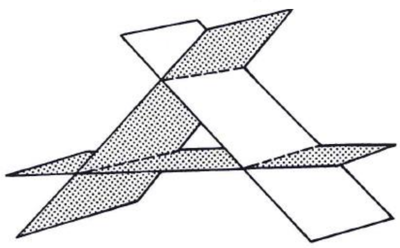
<u>Case 2 :- </u>An infinite number of solutions- 3 planes have a common line(spine of a book), all 3 equations represent the same plane or 2 of the planes are the same with the other crossing (ie. a common line again)

The geometrical interpretation of these two possibilities is as follows:

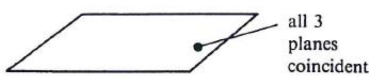
(a) *No solutions* (i.e. inconsistent equations).
This situation will arise when
1. two or more of the planes are parallel (but not coincident). In such cases there is no point common to all three planes.



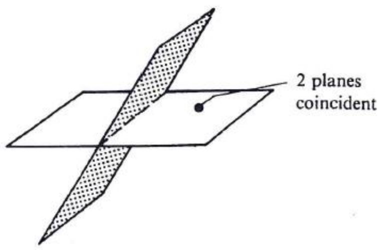
or 2. each plane is parallel to the line of intersection of the other two. Again there will be no point that is common to all three planes.



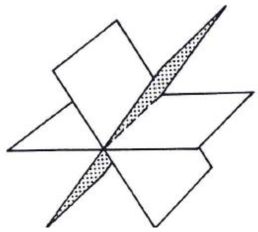
(b) *An infinite number of solutions.*
This situation will arise when
1. all three equations represent the same plane. (i.e. all 3 planes are coincident). Any point in the plane will provide a solution to the equation.



or 2. two of the three planes are coincident and the third plane is not parallel to these two. The planes will intersect in a line and any point on the line will provide a solution.



or 3. the three planes have a common line and any point on this line provides a solution to the equation.



Example

Solve the equations

$2x - 3y + 4z = 1$

$3x - y = 2$

$x + 2y - 4z = 1$

Solution

Example

Solve the equations

$x - 3y - 2z = 9$

$x + 11y + 5z = -5$

$2x + 8y + 3z = 4$

Solution

Homogeneous Equations

$a_1x + b_1y + c_1z = 0$

$a_2x + b_2y + c_2z = 0$

$a_3x + b_3y + c_3z = 0$

$M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

If $M \neq 0$ then there exists M^{-1}

$M^{-1}M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$I\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ therefor if M^{-1} exists i.e. $|M| \neq 0$ then the solutions are trivial.

So in the homogeneous case Non-trivial solution only exist if $|M| = 0$.

Example

Find the value of *k* for which the following equations have a non-trivial solution and find the solution in that case.

$x + 2y = 0$

$3x + ky - z = 0$

$2x + 5y - 2z = 0$

Solution

*System of Linear Equations question sheet

Roots of Polynomials

A quadratic equation of the form $ax^2 + bx + c = 0$, $x \in \mathbb{C}$, where a, b and c are real constants, can have two real roots, one repeated (real) root or two complex roots.

If the roots of this equation are α and β, you can determine the relationship between the **coefficients** of the terms in the quadratic equation and the values of α and β:

$$\begin{aligned} ax^2 + bx + c &= a(x-\alpha)(x-\beta) \\ &= a(x^2 - \alpha x - \beta x + \alpha\beta) \\ &= ax^2 - a(\alpha+\beta)x + a\alpha\beta \end{aligned}$$

Write the quadratic expression in factorised form, then rearrange into the form $ax^2 + bx + c$.

So $b = -a(\alpha + \beta)$ and $c = a\alpha\beta$.

- **If α and β are roots of the equation $ax^2 + bx + c = 0$, then:**
  - $\alpha + \beta = -\dfrac{b}{a}$
  - $\alpha\beta = \dfrac{c}{a}$

Note The sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. Note that these values are real even if the roots are complex, because the sum or product of a conjugate pair is real.

Example 1

The roots of the quadratic equation $2x^2 - 5x - 4 = 0$ are α and β. Without solving the equation, find the values of:

a $\alpha + \beta$ **b** $\alpha\beta$ **c** $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **d** $\alpha^2 + \beta^2$

Problem-solving

Write each expression in terms of $\alpha + \beta$ and $\alpha\beta$:
$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta \Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

Example 2

The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{3}{2}$ and $\beta = \frac{5}{4}$

Find integer values for a, b and c.

*Exercise 4A

Roots of a cubic equation

A cubic equation of the form $ax^3 + bx^2 + cx + d = 0$, $x \in \mathbb{C}$, where a, b, c and d are real constants, will always have at least one real root. It will also have either two further real roots, one further repeated (real) root or two complex roots.

If the roots of this equation are α, β and γ, you can determine the relationship between the coefficients of the terms in the cubic equation and the values of α, β and γ:

$$ax^3 + bx^2 + cx + d = a(x - \alpha)(x - \beta)(x - \gamma)$$
$$= a(x^3 - \alpha x^2 - \beta x^2 - \gamma x^2 + \alpha\beta x + \beta\gamma x + \gamma\alpha x - \alpha\beta\gamma)$$
$$= ax^3 - a(\alpha + \beta + \gamma)x^2 + a(\alpha\beta + \beta\gamma + \gamma\alpha)x - a\alpha\beta\gamma$$

So $b = -a(\alpha + \beta + \gamma)$, $c = a(\alpha\beta + \beta\gamma + \gamma\alpha)$ and $d = -a\alpha\beta\gamma$.

- **If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:**
  - $\alpha + \beta + \gamma = -\frac{b}{a}$
  - $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
  - $\alpha\beta\gamma = -\frac{d}{a}$

Note As with the rule for quadratic equations, the sum of the roots is $-\frac{b}{a}$, and the sum of the products of all possible pairs of roots is $\frac{c}{a}$

Example 3

α, β and γ are the roots of the cubic equation $2x^3 + 3x^2 - 4x + 2 = 0$. Without solving the equation, find the values of:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta + \beta\gamma + \gamma\alpha$ **c** $\alpha\beta\gamma$ **d** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

Example 4

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 - 2i$, $\beta = 1 + 2i$ and $\gamma = 2$. Find integer values for a, b, c and d.

<u>*Exercise 4B</u>

Roots of a quartic equation

Consider the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, $x \in \mathbb{C}$, where a, b, c, d and e are real numbers. If the roots of the equation are α, β, γ and δ, you can determine the relationship between the coefficients of the terms in the equation and the values of α, β, γ and δ:

$$\begin{aligned} ax^4 + bx^3 + cx^2 + dx + e &= a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \\ &= a(x^4 - \alpha x^3 - \beta x^3 - \gamma x^3 - \delta x^3 + \alpha\beta x^2 + \beta\gamma x^2 + \gamma\alpha x^2 + \gamma\delta x^2 + \alpha\delta x^2 \\ &\qquad + \beta\delta x^2 - \alpha\beta\gamma x - \alpha\beta\delta x - \alpha\gamma\delta x - \beta\gamma\delta x + \alpha\beta\gamma\delta) \\ &= ax^4 - a(\alpha + \beta + \gamma + \delta)x^3 + a(\alpha\beta + \beta\gamma + \gamma\alpha + \gamma\delta + \alpha\delta + \beta\delta)x^2 \\ &\qquad - a(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + a\alpha\beta\gamma\delta \end{aligned}$$

So $b = -a(\alpha + \beta + \gamma + \delta)$, $c = a(\alpha\beta + \beta\gamma + \gamma\alpha + \gamma\delta + \alpha\delta + \beta\delta)$, $d = -a(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$ and $e = a\alpha\beta\gamma\delta$.

■ **If α, β, γ and δ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then:**

- $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$
- $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
- $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
- $\alpha\beta\gamma\delta = \frac{e}{a}$

Notation You can use the following abbreviations for these results in your working:

$\Sigma\alpha = -\frac{b}{a}$ $\quad \Sigma\alpha\beta = \frac{c}{a}$ $\quad \Sigma\alpha\beta\gamma = -\frac{d}{a}$

Example 5

The equation $x^4 + 2x^3 + px^2 + qx - 60 = 0$, $x \in \mathbb{C}$, $p, q \in \mathbb{R}$, has roots α, β, γ and δ. Given that $\gamma = -2 + 4\text{i}$ and $\delta = \gamma^*$,

a show that $\alpha + \beta - 2 = 0$ and that $\alpha\beta + 3 = 0$.

b Hence find all the roots of the quartic equation and find the values of p and q.

*Exercise 4C

Expressions relating to the roots of a polynomial

You have already seen several results for finding the values of expressions relating to the roots of a polynomial.

- **The rules for reciprocals:**
  - **Quadratic:** $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$
  - **Cubic:** $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
  - **Quartic:** $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$
- **The rules for products of powers:**
  - **Quadratic:** $\alpha^n\times\beta^n=(\alpha\beta)^n$
  - **Cubic:** $\alpha^n\times\beta^n\times\gamma^n=(\alpha\beta\gamma)^n$
  - **Quartic:** $\alpha^n\times\beta^n\times\gamma^n\times\delta^n=(\alpha\beta\gamma\delta)^n$

In addition to these you have also used the following results for the roots of quadratic equations:

- $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
- $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

There are equivalent results to these for the roots of cubic and quartic equations.

Example 6

a Expand $(\alpha+\beta+\gamma)^2$.

b A cubic equation has roots α, β, γ such that $\alpha\beta+\beta\gamma+\gamma\alpha=7$ and $\alpha+\beta+\gamma=-3$. Find the value of $\alpha^2+\beta^2+\gamma^2$.

You can find an expression for the sum of the squares of a quartic equation in a similar way, by multiplying out $(\alpha+\beta+\gamma+\delta)^2$.

- **The rules for sums of squares:**
  - **Quadratic:** $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
  - **Cubic:** $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$
  - **Quartic:** $\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

Note If you learn these you can use them without proof in your exam.

You can find a similar result for the sum of the cubes of a cubic equation by multiplying out $(\alpha+\beta+\gamma)^3$.

- **The rules for sums of cubes:**
  - **Quadratic:** $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
  - **Cubic:** $\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)+3\alpha\beta\gamma$

Note The result for the sum of cubes for a quartic equation is not required.

*Exercise 4D

Linear transformations of roots

Given the sums and products of the roots of a polynomial, it is possible to find the equation of a second polynomial whose roots are a linear transformation of the roots of the first.

For example, if the roots of a cubic equation are α, β and γ, you need to be able to find the equation of a polynomial with roots $(\alpha + 2)$, $(\beta + 2)$ and $(\gamma + 2)$, or 3α, 3β and 3γ.

Example 8

The cubic equation $x^3 - 2x^2 + 3x - 4 = 0$ has roots α, β and γ.
Find the equations of the polynomials with roots:

a 2α, 2β and 2γ **b** $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$

Problem-solving

Find the sum $\Sigma\alpha$, the pair sum $\Sigma\alpha\beta$ and the product $\alpha\beta\gamma$ for the original equation. Then use these values to find the equivalent sums and products for an equation with roots 2α, 2β and 2γ.

Example 9

The quartic equation $x^4 - 3x^3 + 15x + 1 = 0$ has roots α, β, γ and δ. Find the equation with roots $(2\alpha + 1)$, $(2\beta + 1)$, $(2\gamma + 1)$ and $(2\delta + 1)$.

*Exercise 4E

*Mixed Exercise

Summary of key points

1 If α and β are roots of the equation $ax^2 + bx + c = 0$, then:

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

2 If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:

- $\alpha + \beta + \gamma = \Sigma\alpha = -\frac{b}{a}$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \Sigma\alpha\beta = \frac{c}{a}$
- $\alpha\beta\gamma = -\frac{d}{a}$

3 If α, β, γ and δ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then:

- $\alpha + \beta + \gamma + \delta = \Sigma\alpha = -\frac{b}{a}$
- $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \Sigma\alpha\beta = \frac{c}{a}$
- $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \Sigma\alpha\beta\gamma = -\frac{d}{a}$
- $\alpha\beta\gamma\delta = \frac{e}{a}$

4 The rules for **reciprocals**:

- Quadratic: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$
- Cubic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$
- Quartic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta}$

5 The rules for **products of powers**:

- Quadratic: $\alpha^n \times \beta^n = (\alpha\beta)^n$
- Cubic: $\alpha^n \times \beta^n \times \gamma^n = (\alpha\beta\gamma)^n$
- Quartic: $\alpha^n \times \beta^n \times \gamma^n \times \delta^n = (\alpha\beta\gamma\delta)^n$

6 The rules for **sums of squares**:

- Quadratic: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- Cubic: $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
- Quartic: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

7 The rules for **sums of cubes**:

- Quadratic: $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- Cubic: $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$

Vectors

A scalar quantity is one which requires only size to describe it.

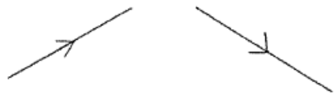
Temperature and distance can be given by a number of the appropriate unit of measurement (21°C, 35 cm).

Some quantities need a direction as well as a number of the appropriate unit of measurement, e.g. 35 km/h north.

A quantity that has both size and direction is called a vector.

Vectors can be represented on diagrams by lines known as "directed line segments".

A "directed line segment" is a line with an arrow:



The length of the line represents the size, the position of the line and the arrow represent the direction. These "directed line segments" have the same length but different directions so they represent different vectors.

Vector Notation

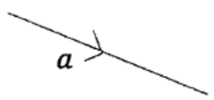
There are 2 common notations:

(i)

We would refer to this vector as: $\overrightarrow{AB}$

In this notation the arrow above AB is always shown from left to right.

(ii)



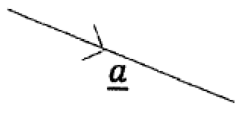
We would refer to this vector as: $\underline{a}$

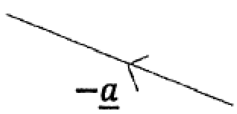
In text books and exams it will be shown in bold as $\boldsymbol{a}$.

In hand-written work you must underline EVERY LETTER representing a VECTOR. It will be marked WRONG in exams if you do not. DO NOT ATTEMPT TO WRITE THEM IN BOLD.

Two vectors are **equal** if they have the same length and are in the same direction.

A vector with the same length as vector $\boldsymbol{a}$ but the opposite direction is the vector $-\boldsymbol{a}$.





Multiplying a vector by a scalar

A scalar is an ordinary number. If the scalar is a positive number then the length of the vector increases but the direction stays the same.

If $\boldsymbol{b} = 2\boldsymbol{a}$ then we say that $\boldsymbol{b}$ is a scalar multiple of $\boldsymbol{a}$.

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction (they are parallel) but different lengths.

We can also see this if we write $\boldsymbol{a}$ and $\boldsymbol{b}$ in column vector form.

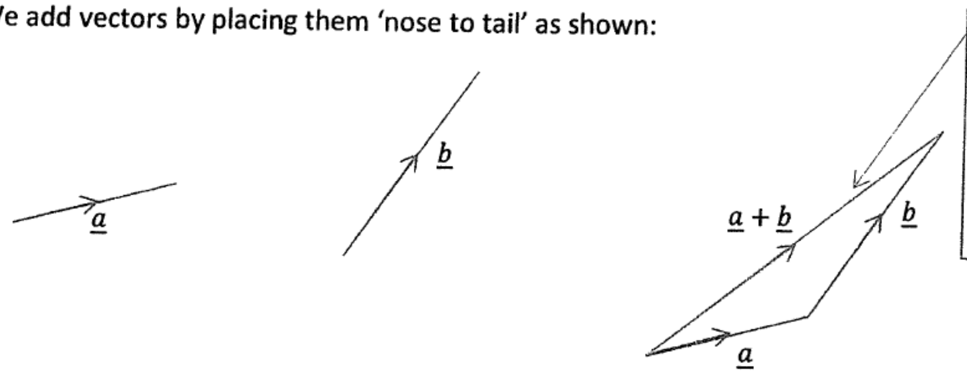
Example

If $\boldsymbol{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$, show that $\boldsymbol{b}$ is a scalar multiple of $\boldsymbol{a}$.

When vectors are in the same direction they are either parallel or in a straight line.

Addition of vectors

We add vectors by placing them 'nose to tail' as shown:



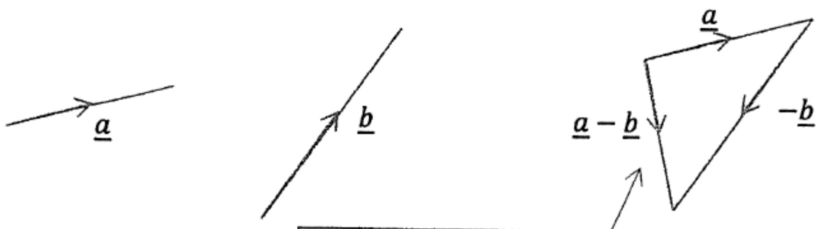
The vector in red is the sum of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Note how it starts at the beginning of $\boldsymbol{a}$ and finishes at the end of $\boldsymbol{b}$

Subtraction of vectors

We do this in a similar way to addition by thinking of 'subtraction' as 'addition of the negative'.

So $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$ gives the following:



The vector in red is the difference of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

When we add or subtract vectors the answer is called the **resultant vector**.

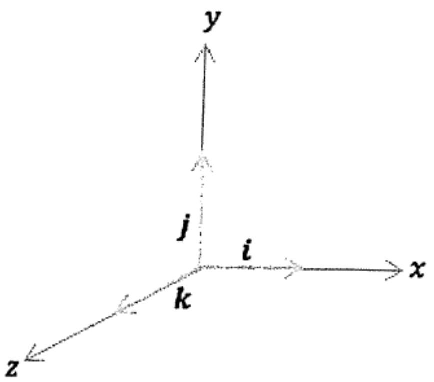
Alternative vector format

Using the standard Cartesian axes we denote:

A vector of length one in the direction of the positive x −axis as $\boldsymbol{i}$.

A vector of length one in the direction of the positive y −axis as $\boldsymbol{j}$.

A vector of length one in the direction of the positive z −axis as $\boldsymbol{k}$.



Every vector can now be represented in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ and these are called the **component vectors** in the x, y and z direction respectively.

Example

Given $\boldsymbol{a} = (3\boldsymbol{i} + 2\boldsymbol{j} + 8\boldsymbol{k})$ and $\boldsymbol{b} = (5\boldsymbol{i} - 6\boldsymbol{j} - 4\boldsymbol{k})$, find the resultant of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Column Vectors

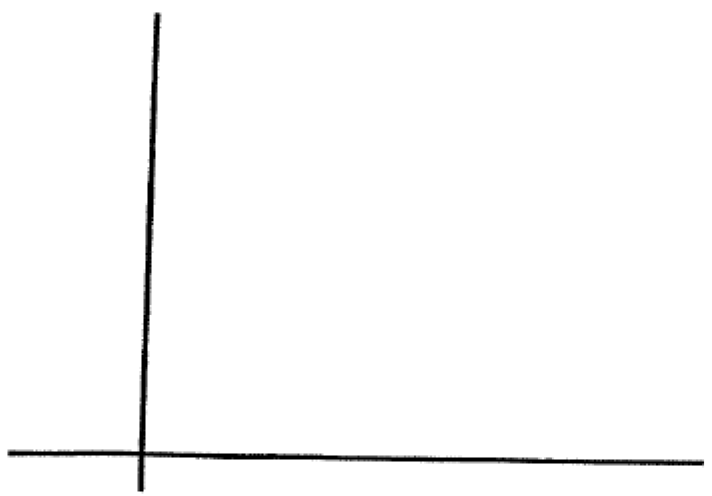
Column vectors are a quicker version of writing vectors. In the above example, we can write

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 5 \\ -6 \\ -4 \end{pmatrix} \qquad \therefore \boldsymbol{a} + \boldsymbol{b} =$$

Position Vectors

A vector $\boldsymbol{a}$ defining the position of a point A in relation to the origin O is called the **position vector** of point A i.e. $\boldsymbol{a} = \overrightarrow{OA}$

Consider point $P(3,2)$, then the position vector $\overrightarrow{OP}$ can be written as



Displacement Vectors

A vector representing movement from point A to point B is called a displacement vector.

Example

Point A has position vector $4\boldsymbol{i} + 3\boldsymbol{j}$ and B has position vector $-2\boldsymbol{i} + 4\boldsymbol{j}$. Find the displacement vector $\overrightarrow{AB}$.

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ where **a** and **b** represent position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively

Magnitude of a Vector

The **magnitude** of a vector represents the vector's size. The magnitude of vector $\boldsymbol{a}$ is denoted by $|\boldsymbol{a}|$. If the vector $\overrightarrow{AB}$ is represented by the line AB then the magnitude is the distance from A to B.

E.g. If $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$ and $\boldsymbol{b} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ then $|\boldsymbol{a}| = \sqrt{x^2 + y^2}$ and $|\boldsymbol{b}| = \sqrt{x^2 + y^2 + z^2}$

N.B. A **unit vector** is a vector of **magnitude 1**.

The unit vector in the direction of vector $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$. It is defined as $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}(\boldsymbol{a}) = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$

Example 1

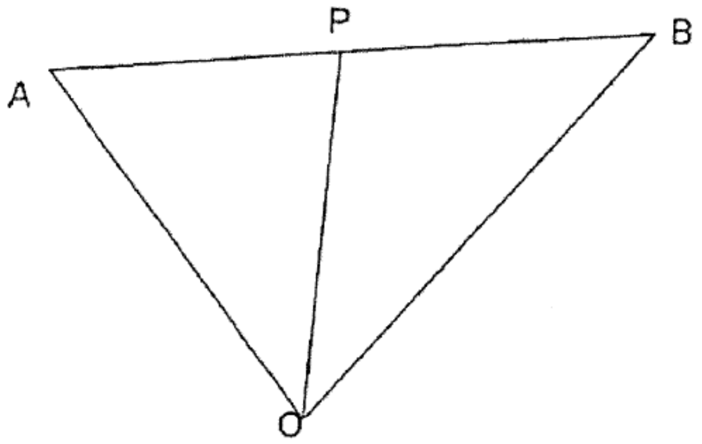
Find a unit vector in the direction of the vector $8\boldsymbol{i} - 6\boldsymbol{j}$.

Example 2

Find a vector of magnitude 14 in the direction of the vector $6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$.

Ratio Theorem

The point P which divides the line AB in the ratio λ : μ has position vector **p** where $\frac{\mu \mathbf{a} + \lambda \mathbf{b}}{\lambda + \mu}$ where **a** and **b** are the position vectors of A and B respectively



Example

If a point A has position vector **i** + 2**j** and B has a position vector 5**i** + **j** find the position vector of the point which divides AB in the ration 1 : -3

(NB the negative sign means the point divides AB externally rather than internally)

Note

The unit vectors ***i*** and ***j*** are base vectors from which other co-planer vectors can be built up. Any pair of non-parallel co-planer vectors could be used instead.

Example

With $\boldsymbol{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ as base vectors, express $\boldsymbol{c} = \begin{pmatrix} 5 \\ 17 \end{pmatrix}$ in the form $\lambda\boldsymbol{a} + \mu\boldsymbol{b}$.

*UPM Ex2B Q3,4,6,7,9,10,11,13,18,19,20

The Scalar Product /Dot Product

The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the product of the magnitudes of the two vectors multiplied by the cosine of the angle between the 2 vectors

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

The scalar product is also referred to as the **dot product**. $|\boldsymbol{a}|, |\boldsymbol{b}|$ and $\cos\theta$ are scalar quantities.

The 'angle between' refers to the angle between the directions of the vectors where these directions are either both towards or both away from the point of intersection i.e.

Note that if θ is acute, the scalar product will be positive. If θ is obtuse, the scalar product will be negative.

The scalar product can also be calculated using components, i.e. if $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then

$$\boldsymbol{a}.\boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

Hence the angle between the two vectors can be calculated:

$$\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Notes

1. The scalar product of 2 perpendicular vectors is zero since $\cos 90 = 0$ i.e. $\boldsymbol{a}.\boldsymbol{b} = 0$.
2. $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$
3. $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}|\cos 0 = |\boldsymbol{a}|^2$
4. $(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c} + \boldsymbol{b}.\boldsymbol{c}$

Example 1

Find the angle between the two vectors $\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ and $2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$.

Example 2

Given that two vectors $\boldsymbol{a}=(3t+1)\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ and $\boldsymbol{b}=(t+3)\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$ are perpendicular, find the possible values of t.

Example 3

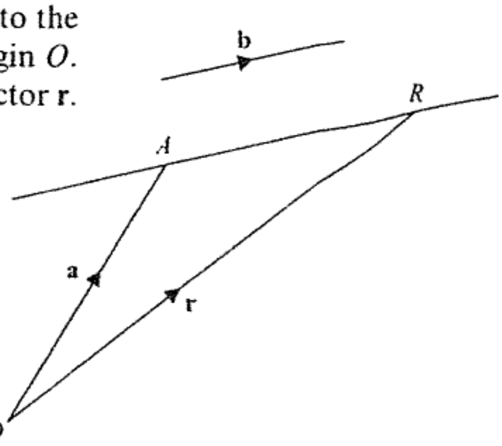
Show that ΔABC is a right angled triangle and find the other two angles given $A(5,3,2)$, $B(2,-1,3)$ and $C(7,-3,10)$.

*UPM Ex2C Q1,3,5,6,8,11,12,15,16

*UPM Ex17A Q1,2,4,6,7,11,12

The vector equation of a straight line

Think of a line that passes through the point A and is parallel to the vector $\mathbf{b}$. The point A has position vector $\mathbf{a}$ referred to the origin O. Let R be any other point on the line and let it have position vector $\mathbf{r}$. Since the line is parallel to $\mathbf{b}$, then:



$$\overrightarrow{AR} = \lambda\mathbf{b}, \text{ where } \lambda \text{ is a scalar}$$

However:

$$\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$$

So: $$\mathbf{r} - \mathbf{a} = \lambda\mathbf{b}$$

or $$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

This is a vector equation of the straight line. The vector $\mathbf{b}$ is in the same direction as the line and is sometimes called the **direction vector of the line**. The vector $\mathbf{a}$ is the position vector of a point on the line and λ is a scalar taking all real values.

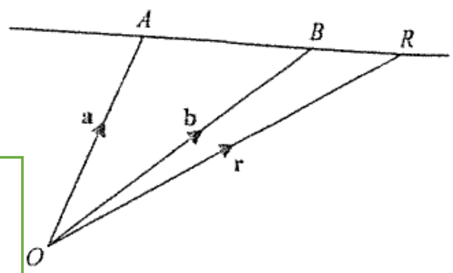
Example

Find a vector equation of the line that passes through the point with position vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and is parallel to the vector $-5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

Example

Find a vector equation of the line that passes through the points A and B with position vectors $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ respectively.

Since the line passes through the points A and B a direction vector for the line is $\overrightarrow{AB}$. (Notice that $\overrightarrow{BA}$ is also a direction vector for the line.) Then:



Example Find the point of intersection of L_1: $\boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + \lambda(\boldsymbol{i} + 3\boldsymbol{j})$ and L_2: $\boldsymbol{r} = 6\boldsymbol{i} - \boldsymbol{j} + \mu(\boldsymbol{i} - 4\boldsymbol{j})$.

Solution

Example Find the vector equation of the line that passes through the point with position vector $2\boldsymbol{i} + 3\boldsymbol{j}$ and is perpendicular to the line $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + \lambda(\boldsymbol{i} - 2\boldsymbol{j})$.

Solution

Example

Find the perpendicular distance from the point A, position vector $\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}$ to

the line L, vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Solution

Definition

Skew lines are lines in 3-D which are not parallel and do not intersect.

Example

Show that the following lines are skew.

line 1: $\mathbf{r} = 17\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} + \lambda(-9\mathbf{i} + 3\mathbf{j} + 9\mathbf{k})$
line 2: $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \mu(6\mathbf{i} + 7\mathbf{j} - \mathbf{k})$

Solution

Cartesian equation of a line in three dimensions

Consider a line that is parallel to the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ and which passes through the point A, position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. If the general point on this line has position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then the vector equation of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Thus $x = a + \lambda p$, $y = b + \lambda q$ and $z = c + \lambda r$ — These are the parametric equations of the line, using the parameter λ.

Isolating λ in each equation gives $\frac{x - a}{p} = \frac{y - b}{q} = \frac{z - c}{r} (= \lambda)$. These are the cartesian equations of the line

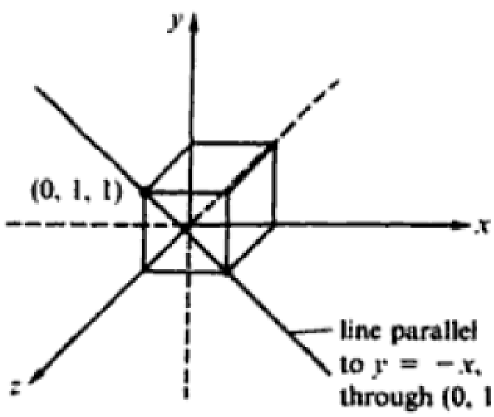
> Thus the line with vector equation $\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ has cartesian
>
> equations: $$\frac{x - a}{p} = \frac{y - b}{q} = \frac{z - c}{r}$$

Notes

1. Given the cartesian equation of a line in the above form it is easy to obtain the vector equation by remembering that the numerator gives the position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ of a point on the line and the denominator gives the direction vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$.
2. It is acceptable to give the cartesian equation of a line in the above form even when one or more of p, q and r are zero.

   For example, the line through (0, 1, 1) and parallel to $y = -x$, i.e. parallel to the vector $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, has vector equation

   $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, (see diagram).



   This gives the parametric equations $\begin{cases} x = -\lambda \\ y = 1 + \lambda \\ z = 1 \end{cases}$

   The cartesian equations of the line can then be written $\frac{x}{-1} = \frac{y - 1}{1} = \frac{z - 1}{0}$ or simply as $\frac{x}{-1} = \frac{y - 1}{1}$, $z = 1$.

Example

Find the cartesian equations of the line that is parallel to the vector **2i** + 3**j** + 4**k** and which passes through the point A, position vector 3**i** − **j** + 2**k**.

Solution

The vector equation of the line is $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$
thus

$$x = 3 + 2\lambda$$
$$y = -1 + 3\lambda$$
$$z = 2 + 4\lambda$$

The cartesian equations are therefore $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-2}{4} (= \lambda)$

Example

The lines L_1 and L_2 are $\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1}$ and $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$. Show that L_1 and L_2 intersect and find the point of intersection.

Solution

*UPM Ex17C Q1-4,6,7,9a,10-13,15-17,19,20ac

The vector product

The scalar product of the vectors **a** and **b** is

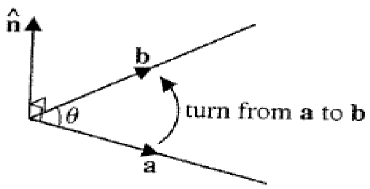
$$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

where θ is the angle between the vectors **a** and **b**

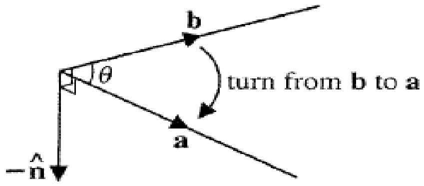
The **vector** (or **cross**) **product** of the vectors **a** and **b** is defined as

■ $$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\hat{\mathbf{n}}$$

Once again θ is the angle between **a** and **b**, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both **a** and **b**. The direction of $\hat{\mathbf{n}}$ is that in which a right-handed corkscrew would move when turned from **a** to **b**:



If the turn is in the opposite sense, that is from **b** to **a**, then the movement of the corkscrew is in the opposite sense to $\hat{\mathbf{n}}$. That is, it is in the direction of $-\hat{\mathbf{n}}$.

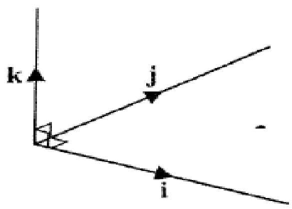


So:

$$\begin{aligned}\mathbf{b} \times \mathbf{a} &= |\mathbf{b}|\,|\mathbf{a}|\,\sin\theta(-\hat{\mathbf{n}}) \\ &= -|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\hat{\mathbf{n}} \\ &= -\mathbf{a} \times \mathbf{b}\end{aligned}$$

■ $$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Be careful, therefore, because $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. The vector product is *not* commutative. Notice that **a.b** is called the scalar product of **a** and **b** because the result is a scalar and that $\mathbf{a} \times \mathbf{b}$ is called the vector product of **a** and **b** because the result is another vector.



Vector product equal to zero

If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then since $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\, \hat{\mathbf{n}}$

either $\quad |\mathbf{a}| = 0 \Rightarrow \mathbf{a} = \mathbf{0}$

or $\quad |\mathbf{b}| = 0 \Rightarrow \mathbf{b} = \mathbf{0}$

or $\quad \sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$

But if either $\theta = 0$ or $\theta = \pi$ then $\mathbf{a}$ and $\mathbf{b}$ are in the same direction (and either in the same sense or in opposite senses). In either case, if $\sin\theta = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are parallel.

So:

- if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then either $\mathbf{a} = \mathbf{0}$, or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel.

Now the vectors **i**, **j** and **k** are such that each one is perpendicular to the other two. Also their relative positions are such that if a right-handed corkscrew were turned from **i** to **j** it would move in the direction **k**. You should also be able to see that if a right-handed corkscrew were turned from **j** to **k** it would move in the direction **i** and if it were turned from **k** to **i**, it would move in the direction **j**.

So $\quad \mathbf{i} \times \mathbf{j} = |\mathbf{i}|\,|\mathbf{j}| \sin 90^\circ\, \mathbf{k}$

$\quad = (1 \times 1 \times 1)\mathbf{k} = \mathbf{k}$

- That is: $\quad \mathbf{i} \times \mathbf{j} = \mathbf{k}$

  Similarly $\quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$

  and $\quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$

and $j \times i = -k$
$k \times j = -i$
$i \times k = -j$

Also, since the angle between **i** and itself is 0 and $\sin 0 = 0$,

- then: $\quad \mathbf{i} \times \mathbf{i} = \mathbf{0}$

  Similarly $\quad \mathbf{j} \times \mathbf{j} = \mathbf{0}$

  and $\quad \mathbf{k} \times \mathbf{k} = \mathbf{0}$

The vector product of $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$= a_1b_1(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + a_1b_3(\mathbf{i} \times \mathbf{k}) + a_2b_1(\mathbf{j} \times \mathbf{i})$

$\quad + a_2b_2(\mathbf{j} \times \mathbf{j}) + a_2b_3(\mathbf{j} \times \mathbf{k}) + a_3b_1(\mathbf{k} \times \mathbf{i}) + a_3b_2(\mathbf{k} \times \mathbf{j}) + a_3b_3(\mathbf{k} \times \mathbf{k})$

$= a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i})$

(because $\mathbf{i} \times \mathbf{k} = -\mathbf{k} \times \mathbf{i} = -\mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{j} \times \mathbf{k} = -\mathbf{i}$)

$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Result

- $$(a_1\mathbf{i} + +a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example

Find a unit vector which is perpendicular to both $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Solution

Example

Find the sine of the acute angle between $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

Solution

*P4 Book page 213 EX6A Q1alt,2,4,5,6,7alt,9,10

Applications of the vector product

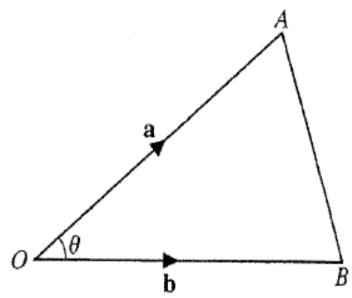
Area of a triangle

You should know that if you are given a triangle ABC then the area of the triangle is given by $\frac{1}{2}ab\sin C$.

If, therefore, you have a triangle OAB, then the area of the triangle is given by

$$\tfrac{1}{2}(OA)(OB)\sin\theta$$

where $\angle AOB = \theta$



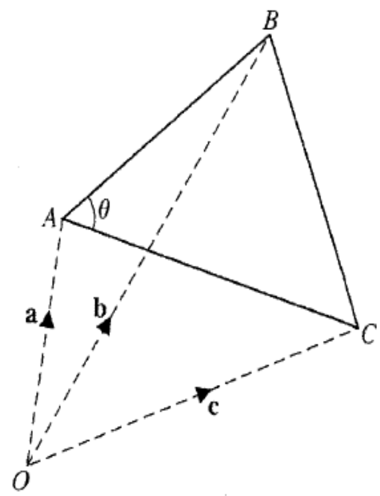
Now if the position vectors of A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$, respectively, then

$$\begin{aligned}\text{area of } \triangle AOB &= \tfrac{1}{2}(OA)(OB)\sin\theta \\ &= \tfrac{1}{2}|\mathbf{a}|\,|\mathbf{b}|\sin\theta \\ &= \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}|\end{aligned}$$

■ Area of $\triangle AOB = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

If you have a triangle ABC where the position vectors of A, B, C relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, then the area of the triangle can be calculated in a similar fashion.

The area of the triangle is given by $\frac{1}{2}(AB)(AC)\sin\theta$, where $\theta = \angle BAC$.



That is, $\frac{1}{2}|\overrightarrow{AB}|\,|\overrightarrow{AC}|\sin\theta$

But $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ so the area of the triangle is

$$\begin{aligned}&\tfrac{1}{2}|\mathbf{b}-\mathbf{a}|\,|\mathbf{c}-\mathbf{a}|\sin\theta \\ &= \tfrac{1}{2}|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})| \\ &= \tfrac{1}{2}|(\mathbf{b}\times\mathbf{c})-(\mathbf{b}\times\mathbf{a})-(\mathbf{a}\times\mathbf{c})+(\mathbf{a}\times\mathbf{a})| \\ &= \tfrac{1}{2}|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}| \\ &= \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|\end{aligned}$$

since $\mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c}$, $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ and $\mathbf{a}\times\mathbf{a} = \mathbf{0}$.

■ Area of $\triangle ABC = \frac{1}{2}|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$

Example

Find the area of the triangle OAB where O is the origin, A has position vector $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and B has position vector $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

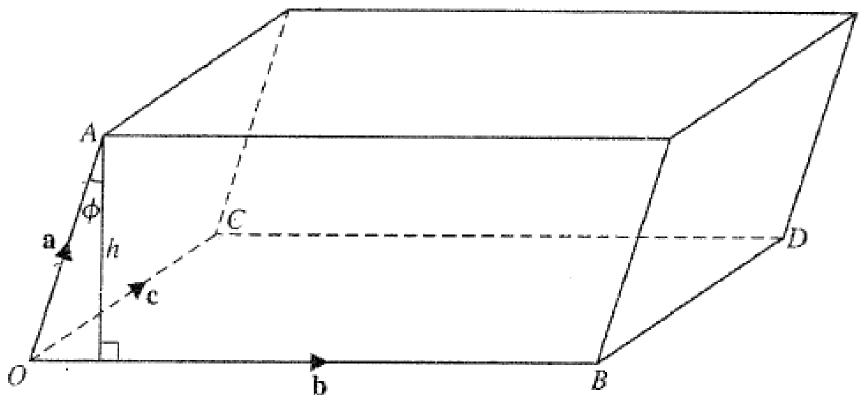
Solution

Example

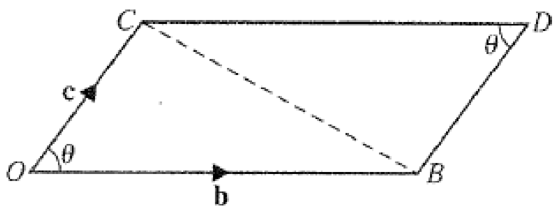
Find the area of the triangle ABC where the position vectors of A, B, C relative to the origin O are $2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ respectively.

Solution

Volume of a parallelepiped



The volume of a parallelepiped is given by (area of base) × h where h is the perpendicular distance between the base and the top face. In the parallelepiped above, O is the origin and A, B, C have position vectors **a**, **b** and **c** respectively. The base of the parallelepiped is a parallelogram.



Now the area of $\triangle OBC$ is $\frac{1}{2}(OB)(OC)\sin\theta$ and the area of $\triangle DCB$ is $\frac{1}{2}(DC)(DB)\sin\theta$. (You should know that the opposite angles of a parallelogram are equal in size.) But because $OBDC$ is a parallelogram, $OC = BD$ and $OB = CD$.

So:

$$\begin{aligned}\text{area of parallelogram } OBDC &= \tfrac{1}{2}(OB)(OC)\sin\theta + \tfrac{1}{2}(DC)(DB)\sin\theta \\ &= \tfrac{1}{2}(OB)(OC)\sin\theta + \tfrac{1}{2}(OB)(OC)\sin\theta \\ &= (OB)(OC)\sin\theta \\ &= |\mathbf{b} \times \mathbf{c}|\end{aligned}$$

The volume of the parallelepiped is therefore $|\mathbf{b} \times \mathbf{c}|h$.

Now $$\frac{h}{OA} = \cos\phi$$

where ϕ is the angle between the vertical and OA.

So: $$h = OA \cos \phi$$

and the volume is $|\mathbf{b} \times \mathbf{c}| OA \cos \phi$

$$= |\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}| \cos \phi$$

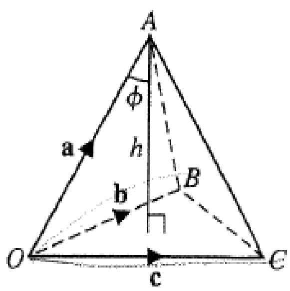
$$= |\mathbf{a}|\,|\mathbf{b} \times \mathbf{c}| \cos \phi$$

But $\mathbf{b} \times \mathbf{c}$ is vertically up, in the direction of h, since $\mathbf{b} \times \mathbf{c}$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$.

So ϕ is the angle between $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$, and $|\mathbf{a}|\,|\mathbf{b} \times \mathbf{c}| \cos \phi$ is the scalar product of $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$.

- Thus the volume of the parallelepiped is given by $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$, which is usually written without the brackets, because there can be no confusion, as simply $\mathbf{a}.\mathbf{b} \times \mathbf{c}$. This quantity is usually known as a **triple scalar product**.

Volume of a tetrahedron



The volume of a tetrahedron is given by the formula $\frac{1}{3}$ (area of base) $\times h$, where h is the perpendicular height.

In the tetrahedron $OABC$ above, you have O as the origin, $\mathbf{a}$ as the position vector of A, $\mathbf{b}$ as the position vector of B and $\mathbf{c}$ as the position vector of C. The perpendicular height makes an angle ϕ with OA.

You know that the area of the triangular base is $\frac{1}{2}|\mathbf{b} \times \mathbf{c}|$.

Also: $$h = OA \cos \phi = |\mathbf{a}| \cos \phi$$

So the volume of the tetrahedron is given by

$$\tfrac{1}{3} \times \tfrac{1}{2} |\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}| \cos \phi$$

But because $\mathbf{b} \times \mathbf{c}$ is in the direction of h, the angle ϕ is the angle between $\mathbf{b} \times \mathbf{c}$ and $\mathbf{a}$.

- Therefore the volume of the tetrahedron is given by

$$\tfrac{1}{6}\mathbf{a}.\mathbf{b} \times \mathbf{c}$$

Evaluating the triple scalar product

You know that $\mathbf{b} \times \mathbf{c}$ can be evaluated as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

So $\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\,\mathbf{i} + (b_3c_1 - b_1c_3)\,\mathbf{j} + (b_1c_2 - b_2c_1)\,\mathbf{k}$ (A)

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then:

$\mathbf{a}.\mathbf{b} \times \mathbf{c} = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$ (B)

However, if you compare (A) and (B) you will see that they are the same, except that in (B) $\mathbf{i}$ is replaced by a_1, $\mathbf{j}$ is replaced by a_2 and $\mathbf{k}$ is replaced by a_3. This leads to the conclusion that

■
$$\mathbf{a}.\mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

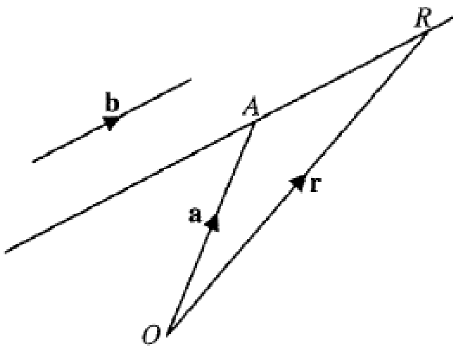
Example

A tetrahedron has vertices at $A(0, 1, 0)$, $B(1, 1, 2)$, $C(-2, 1, 3)$ and $D(2, 0, 1)$ relative to the origin O. Find the volume of the tetrahedron.

Solution

*P4 book page 220 Ex6B Q1,3,5-11,13,14,16

The equation of a straight line



If a line is parallel to a vector **b**, and if a point A on the line has position vector **a** and any other point R on the line has position vector **r**, then an equation of the line is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where λ is a scalar parameter (Book P3, page 138).

Now the vector $\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$ and $\overrightarrow{AR}$ is parallel to **b**. But you learned earlier (page 211) that if two vectors are parallel, then their vector product is zero.

- **So another form of the equation of the straight line is**

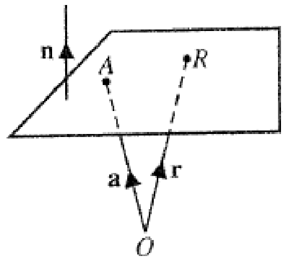
$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$$

Example

Find an equation of the straight line which passes through point $A(2, -1, 1)$ and is parallel to $\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$.

Solution

The scalar product form of the equation of a plane



The point A, with position vector **a** lies in a given plane. The vector **n** is perpendicular to the plane. The point R with position vector **r** is any other point in the plane.

Then: $\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$

Since both A and R lie in the given plane, then $\overrightarrow{AR}$ must lie in the plane. But if **n** is perpendicular to the plane then the directions of **n** and $\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$ must be perpendicular. However, if two vectors are perpendicular then their scalar product is zero (see Book P3, page 132).

So: $(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$

or $\mathbf{r}.\mathbf{n} - \mathbf{a}.\mathbf{n} = 0$

$\Rightarrow \quad \mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$

If $\mathbf{a}.\mathbf{n} = p$ then an equation of the plane is $\mathbf{r}.\mathbf{n} = p$.

- Given that a plane contains a point with position vector **a**, that **r** is the position vector of any other point in the plane and that **n** is perpendicular to the plane, the scalar product form of the equation of the plane is $\mathbf{r}.\mathbf{n} = p$, where $p = \mathbf{a}.\mathbf{n}$.

<u>Cartesian equation of a Plane</u>

Consider a plane with equation $r.n = \rho$ where $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Writing ***r***, the position vector of the general point as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Gives $\begin{pmatrix} x \\ y \\ z \end{pmatrix} . \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \rho$

i.e. $ax + by + cz = \rho$

Thus a plane perpendicular to $ai + bj + ck$ has Cartesian equation $ax + by + cz = \rho$

<u>Note</u>

To show that a line lies in a given plane, show either:-

- Line and plane are parallel AND They have a common point
  OR
- The line and plane contain two common points

Example

The point A with position vector $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ lies in a plane. The vector $-\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to the plane. Find an equation of the plane

(a) in scalar product form

(b) in cartesian form.

Solution

Example

Show that the line with equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j})$ where λ is a scalar parameter lies in the plane with equation

r.(i-2j+2k)=-1.

Solution

The vector equation of a plane

If A lies in the given plane and has position vector $\mathbf{a}$ and R is any point in the plane and has position vector $\mathbf{r}$, and $\mathbf{b}$ and $\mathbf{c}$ are two non-parallel vectors in the plane, neither of which is zero, then

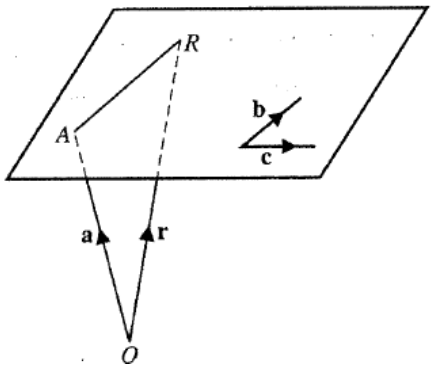
$$\mathbf{r} = \mathbf{a} + \overrightarrow{AR}$$

But $\overrightarrow{AR}$, since it lies in the plane, can be written:

$$\overrightarrow{AR} = \lambda\mathbf{b} + \mu\mathbf{c}$$

where λ, μ are scalar parameters.

So: $$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$



- The vector equation of a plane where $\mathbf{a}$ is the position vector of a point in the plane and $\mathbf{b}$ and $\mathbf{c}$ are non-parallel vectors in the plane, neither of which is zero, is given by

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}, \quad \lambda, \mu \text{ scalars}$$

Example

Three points in a plane have coordinates $A(1, -1, 0)$, $B(0, 1, 1)$ and $C(2, 1, -2)$ referred to an origin O. Find a vector equation of the plane.

Solution

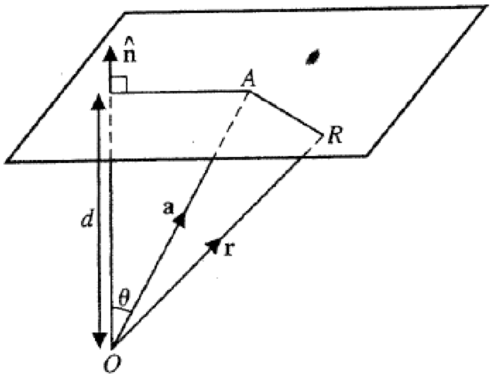
Example

Find a cartesian equation of the plane containing the points $A(1, 1, 1)$, $B(2, 1, 0)$ and $C(2, 2, -1)$.

Solution

*P4 book page 228 Ex6C Q1ab,2ab,3ab,4ab,5ab,6a,7ac,8ac,9-12

Distance of a point from a plane



The diagram shows a plane which contains the point A, with position vector $\mathbf{a}$, and also contains any other point R, with position vector $\mathbf{r}$. The unit vector $\hat{\mathbf{n}}$ is perpendicular to the plane. The line OA makes an angle θ with $\hat{\mathbf{n}}$ and d is the distance of O from the plane.

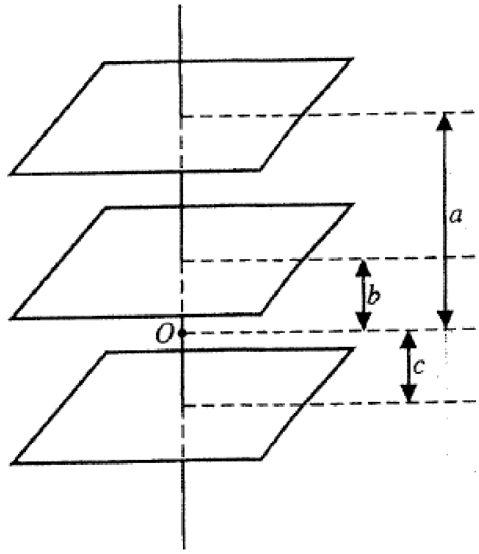
Then: $d = a\cos\theta$

$= a.1.\cos\theta$

$= |\mathbf{a}|\,|\hat{\mathbf{n}}|\cos\theta$, since $\hat{\mathbf{n}}$ is a unit vector

So $d = \mathbf{a}.\hat{\mathbf{n}}$

Now an equation of the plane is $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$. Thus if you replace the vector $\mathbf{n}$ by $\hat{\mathbf{n}}$ in the scalar product form of the equation of the plane you get $\mathbf{r}.\hat{\mathbf{n}} = d$ where d is the perpendicular distance of the origin from the plane.



Consider 3 parallel planes Π_1, Π_2 and Π_3 with vector equations

$\underset{\sim}{r}.\underset{\sim}{\hat{n}} = 3$ Π_1 will be a distance 3 units from the origin.

$\underset{\sim}{r}.\underset{\sim}{\hat{n}} = 1$ Π_2 " " 1 " ".

$\underset{\sim}{r}.\underset{\sim}{\hat{n}} = -2$ Π_3 " " 2 " ".

and Π_3 is on the other side of the origin from Π_1 and Π_2.

$\Rightarrow$ Π_1 is 2 units from Π_2 and Π_1 is 5 units from Π_3.

Example

Find the perpendicular distance of the origin from the plane with equation $\mathbf{r}.(2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}) = 7$.

Solution

Example

Find the distance of the point (3, 1, 6) from the plane with equation $\mathbf{r}.(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 13$.

Solution

The line of intersection of two planes

In general when two planes intersect their intersection is a straight line. If you can write the equation of each plane in cartesian form, then by solving the two equations simultaneously you should be able to obtain an equation of the line of intersection, as the next example shows.

Example

Find, in vector form, an equation of the line of intersection of the plane $\mathbf{r}.(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 3$ with the plane $\mathbf{r}.(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 5$.

Solution

$\mathbf{r}.(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 3$ can be written:

$$x + y + z = 3 \qquad (1)$$

and $\mathbf{r}.(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 5$ can be written:

$$x + 2y + 3z = 5 \qquad (2)$$

(2) – (1) gives $\quad y + 2z = 2$

So: $\quad y = 2 - 2z$

Substituting in (1) gives:

$$x + 2 - 2z + z = 3$$

$$\Rightarrow \quad x - z = 1$$

or $\quad x = 1 + z$

If you let $z = \lambda$, say, then

$$\frac{x-1}{1} = \frac{2-y}{2} = \frac{z}{1} (= \lambda)$$

which is an equation of the line of intersection in cartesian form.

So: $\quad x = 1 + \lambda,\ y = 2 - 2\lambda,\ z = \lambda$

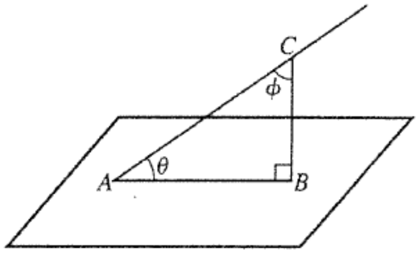
If $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

then: $\quad \mathbf{r} = (1+\lambda)\mathbf{i} + (2-2\lambda)\mathbf{j} + \lambda\mathbf{k}$

That is: $\quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

which is an equation of the line of intersection in vector form.

The angle between a line and a plane



The angle between a line and a plane is the angle between the line and its projection on the plane (see Book P2, page 188). In the diagram, AC is the line and AB is its projection on the plane. So you have to find the angle $CAB = \theta$.

But $\theta = 90 - \emptyset$ is the angle between the line BC i.e. $\emptyset$ is the angle between AC and a normal to the plane.

*So, if asked to find the angle between a line and a plane, first find the angle between the line and the normal to the plane and then subtract the answer from 90°. This is the required angle.

Example

Find the acute angle between the line with equations

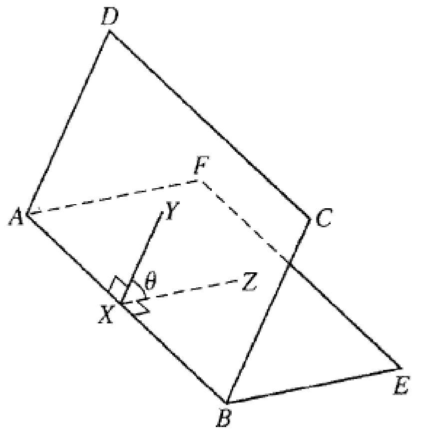
$$\frac{x+1}{2} = y - 2 = \frac{z-3}{-2} = \lambda$$

and the plane with equation

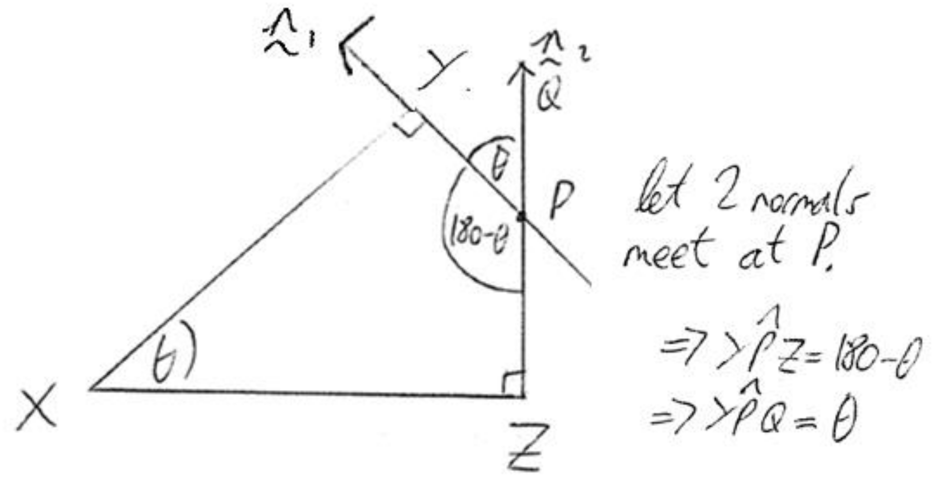
$$2x + 3y - 7z = 5$$

Solution

Angle between two planes



The angle between two planes is the angle between two perpendiculars (one in each plane) drawn from a point on their line of intersection (Book P2, page 191). If you are given vector equations of the two planes you need to be able to calculate $\angle YXZ = \theta$. So from Y you draw the perpendicular to the plane $ABCD$ and from Z you draw the perpendicular to the plane $ABEF$.



<u>Result:-</u> The angle between two planes is the angle between the two normal.

<u>Example</u> Find the angle between the planes with equations

$$\mathbf{r}.(2\mathbf{i}+\mathbf{j}+3\mathbf{k}) = 5$$

and

$$\mathbf{r}.(2\mathbf{i}+3\mathbf{j}+\mathbf{k}) = 7$$

<u>Solution</u>

*P4 book page 236 Ex6D Q1ac,2ac,3,4ab,5ab,6,7,8ab,11,13-17,19,20,21

Complex Numbers

The quadratic equation $ax^2 + bx + c = 0$ has solutions given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the expression under the square root is negative, there are no real solutions.

You can find solutions to the equation in all cases by extending the number system to include $\sqrt{-1}$. Since there is no real number that squares to produce -1, the number $\sqrt{-1}$ is called an **imaginary number**, and is represented using the letter **i**. Sums of real and imaginary numbers, for example $3 + 2\text{i}$, are known as **complex numbers**.

- **$\text{i} = \sqrt{-1}$**
- **An imaginary number is a number of the form $b\text{i}$, where $b \in \mathbb{R}$.**
- **A complex number is written in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.**

Notation The set of all complex numbers is written as $\mathbb{C}$.
For the complex number $z = a + b\text{i}$:
- $\text{Re}(z) = a$ is the real part
- $\text{Im}(z) = b$ is the imaginary part

In a complex number, the real part and the imaginary part cannot be combined to form a single term.

- **Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.**
- **You can multiply a real number by a complex number by multiplying out the brackets in the usual way.**

Example

Simplify each of the following, giving your answers in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.

a $(2 + 5\text{i}) + (7 + 3\text{i})$ **b** $(2 - 5\text{i}) - (5 - 11\text{i})$ **c** $2(5 - 8\text{i})$ **d** $\frac{10 + 6\text{i}}{2}$

Solution

You can use complex numbers to find solutions to any quadratic equation with real coefficients.

- **If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which are real.**

Example

Solve the equation $z^2 + 9 = 0$.

Solution

Example

Solve the equation $z^2 + 6z + 25 = 0$.

Solution

Multiplying complex numbers

You can multiply complex numbers using the same technique that you use for multiplying brackets in algebra. You can use the fact that $i = \sqrt{-1}$ to simplify powers of i.

- $\mathbf{i^2 = -1}$

Example

Express each of the following in the form $a + bi$, where a and b are real numbers.

a $(2 + 3i)(4 + 5i)$ **b** $(7 - 4i)^2$

Solution

Complex conjugation

- **For any complex number $z = a + bi$, the complex conjugate of the number is defined as $z^* = a - bi$.**

Notation Together z and z^* are called a **complex conjugate pair**.

Example

Given that $z = 2 - 7i$,

a write down z^* **b** find the value of $z + z^*$ **c** find the value of zz^*

Solution

Note The modulus of a complex number is denoted by $|z|$ $where$ $|z| = \sqrt{a^2 + b^2}$ i.e. $zz^* = |z|^2$.

Dividing Complex Numbers

you multiply both the numerator and the denominator by the complex conjugate of the denominator and then simplify the result.

Links The method used to divide complex numbers is similar to the method used to rationalise a denominator when simplifying surds.

Example

Write $\frac{5 + 4i}{2 - 3i}$ in the form $a + bi$.

Solution

Note two complex numbers are equal if and only if their real and imaginary parts are separately equal.

e.g. If $z_1 = z_2$

$$\therefore x_1 + iy_1 = x_2 + iy_2$$

$$\therefore (x_1 - x_2)^2 = [i(y_2 - y_1)]^2$$

$$\therefore (x_1 - x_2)^2 = -1(y_2 - y_1)^2$$

$$\therefore (x_1 - x_2)^2 + (y_2 - y_1)^2 = 0$$

$\therefore (x_1 - x_2)^2 = 0$ and $(y_2 - y_1)^2 = 0$ (since the square of real numbers must be positive)

$$\therefore x_1 = x_2 \text{ and } y_1 = y_2$$

Note If $x + iy = 0$ then $x = 0$ and $y = 0$

***P3 book page 166 EX7A Q1ace,2ace,3ace,4,5ace,7ace,8ace,9ace,11,12**

Finding the square root of a complex number

Example Find $\sqrt{3+4i}$

Solution

***P3 book page 179 EX7C Q1ace,2ace,3,4,8**

Roots of quadratic equations

- **For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are non-real complex numbers, then they occur as a conjugate pair.**

Another way of stating this is that for a real-valued quadratic function f(z), if z_1 is a root of f(z) = 0 then z_1* is also a root. You can use this fact to find one root if you know the other, or to find the original equation.

- **If the roots of a quadratic equation are α and β, then you can write the equation as $(z - \alpha)(z - \beta) = 0$**

  or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$

Notation Roots of complex-valued polynomials are often written using Greek letters such as α (alpha), β (beta) and γ (gamma).

Example

Given that $\alpha = 7 + 2\text{i}$ is one of the roots of a quadratic equation with real coefficients,

a state the value of the other root, β

b find the quadratic equation

c find the values of $\alpha + \beta$ and $\alpha\beta$ and interpret the results.

Solution

Solving cubic and quartic equations

You can generalise the rule for the roots of quadratic equations to any polynomial with real coefficients.

- **If f(z) is a polynomial with real coefficients, and z_1 is a root of f(z) = 0, then z_1* is also a root of f(z) = 0.**

Note If z_1 is real, then $z_1^* = z_1$.

You can use this property to find roots of cubic and quartic equations with real coefficients.

- **An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.**
- **For a cubic equation with real coefficients, either:**
  - **all three roots are real, or**
  - **one root is real and the other two roots form a complex conjugate pair.**

Watch out A real-valued cubic equation might have two, or three, repeated real roots.

Example

Given that -1 is a root of the equation $z^3 - z^2 + 3z + k = 0$,

a find the value of k **b** find the other two roots of the equation.

Solution

- **An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.**
- **For a quartic equation with real coefficients, either:**
  - **all four roots are real, or**
  - **two roots are real and the other two roots form a complex conjugate pair, or**
  - **two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.**

Watch out A real-valued quartic equation might have repeated real roots or repeated complex roots.

Example

Given that $3 + \mathrm{i}$ is a root of the quartic equation $2z^4 - 3z^3 - 39z^2 + 120z - 50 = 0$, solve the equation completely.

Solution ***use different method. Divide through by quadratic to reduce quartic to quadratic then solve that quadratic***

Example

Show that $z^2 + 4$ is a factor of $z^4 - 2z^3 + 21z^2 - 8z + 68$.

Hence solve the equation $z^4 - 2z^3 + 21z^2 - 8z + 68 = 0$.

Solution

*Edexcel Book E1F and Mixed Exercise 1 as extra

Summary of key points

1. $\mathbf{i} = \sqrt{-1}$ and $\mathbf{i}^2 = -1$

2. An **imaginary number** is a number of the form $b\text{i}$, where $b \in \mathbb{R}$.

3. A **complex number** is written in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.

4. Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.

5. You can multiply a real number by a complex number by multiplying out the brackets in the usual way.

6. If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which is real.

7. For any complex number $z = a + b\text{i}$, the **complex conjugate** of the number is defined as $z^* = a - b\text{i}$.

8. For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are non-real complex numbers, then they occur as a conjugate pair.

9. If the roots of a quadratic equation are α and β, then you can write the equation as $(z - \alpha)(z - \beta) = 0$ or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$.

10. If $\text{f}(z)$ is a polynomial with real coefficients, and z_1 is a root of $\text{f}(z) = 0$, then $z_1{}^*$ is also a root of $\text{f}(z) = 0$.

11. An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.

    For a cubic equation with real coefficients, either:

    - all three roots are real, or
    - one root is real and the other two roots form a complex conjugate pair.

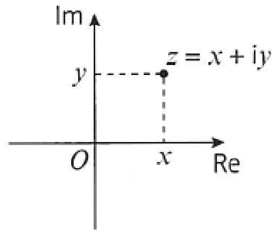
12. An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.

    For a quartic equation with real coefficients, either:

    - all four roots are real, or
    - two roots are real and the other two roots form a complex conjugate pair, or
    - two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.

The Argand Diagram

- **You can represent complex numbers on an Argand diagram. The x-axis on an Argand diagram is called the real axis and the y-axis is called the imaginary axis. The complex number $z = x + \mathrm{i}y$ is represented on the diagram by the point $P(x, y)$, where x and y are Cartesian coordinates.**



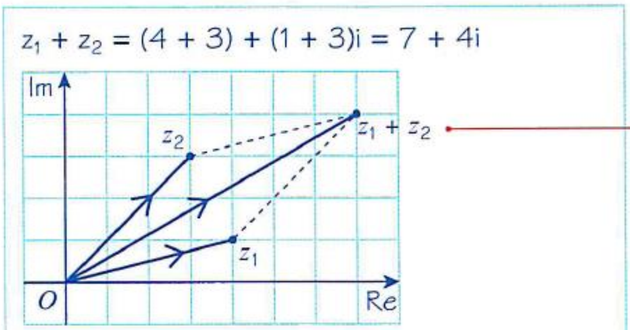
Complex numbers can also be represented as vectors on the Argand diagram.

- **The complex number $z = x + \mathrm{i}y$ can be represented as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on an Argand diagram.**

You can add or subtract complex numbers on an Argand diagram by adding or subtracting their corresponding vectors.

Example

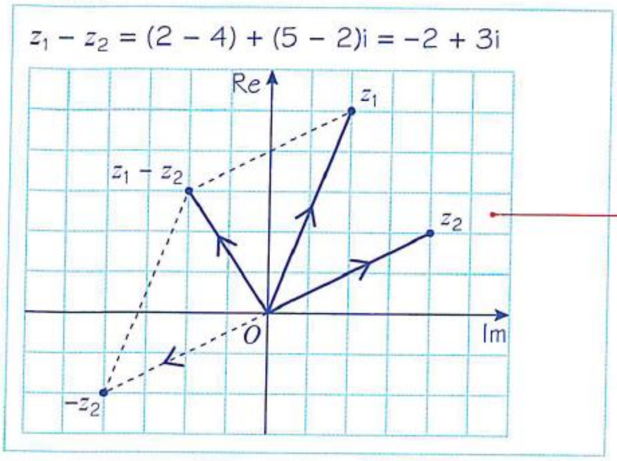
$z_1 = 4 + \mathrm{i}$ and $z_2 = 3 + 3\mathrm{i}$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram.



The vector representing $z_1 + z_2$ is the diagonal of the parallelogram with vertices at O, z_1 and z_2. You can use vector addition to find $z_1 + z_2$:
$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

Example

$z_1 = 2 + 5\mathrm{i}$ and $z_2 = 4 + 2\mathrm{i}$. Show z_1, z_2 and $z_1 - z_2$ on an Argand diagram.



The vector corresponding to z_2 is $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$, so the vector corresponding to $-z_2$ is $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$.

The vector representing $z_1 - z_2$ is the diagonal of the parallelogram with vertices at O, z_1 and $-z_2$.

Modulus and argument

The **modulus** or absolute value of a complex number is the magnitude of its corresponding vector.

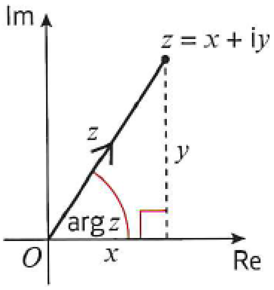
- **The modulus of a complex number, $|z|$, is the distance from the origin to that number on an Argand diagram. For a complex number $z = x + iy$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$.**

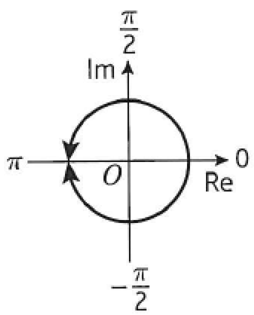
Notation The modulus of the complex number z is written as r, $|z|$ or $|x + iy|$.

The **argument** of a complex number is the angle its corresponding vector makes with the positive real axis.

- **The argument of a complex number, arg z, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram, measured in an anticlockwise direction. For a complex number $z = x + iy$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$**

Notation The argument of the complex number z is written as arg z. It is usually given in radians, where
- 2π radians = 360°
- π radians = 180°





The argument θ of any complex number is usually given in the range $-\pi < \theta \leqslant \pi$. This is sometimes referred to as the **principal argument**.

Example

$z = 2 + 7i$, find:

a the modulus of z

b the argument of z, giving your answer in radians to 2 decimal places.

Solution

Example

$z = -4 - \text{i}$, find:

a the modulus of z

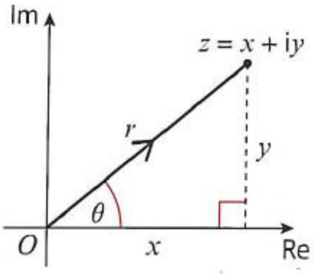
b the argument of z, giving your answer in radians to 2 decimal places.

Solution

Modulus–argument form of complex numbers

You can write any complex number in terms of its modulus and argument.

- **For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus–argument form of z is $z = r(\cos\theta + \text{i}\sin\theta)$**



From the right-angled triangle, $x = r\cos\theta$ and $y = r\sin\theta$.

$z = x + \text{i}y = r\cos\theta + \text{i}r\sin\theta = r(\cos\theta + \text{i}\sin\theta)$

This formula works for a complex number in any quadrant of the Argand diagram. The argument, θ, is usually given in the range $-\pi < \theta \leqslant \pi$, although the formula works for any value of θ measured anticlockwise from the positive real axis.

Example

Express $z = -\sqrt{3} + \text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

*Edexcel Book Ex2A Q2-6

*Edexcel Book Ex2C Q1-3(alt parts),Q4-6

You can use the following rules to multiply complex numbers quickly when they are given in modulus–argument form.

- **For any two complex numbers z_1 and z_2,**
  - $|z_1 z_2| = |z_1||z_2|$
  - **$\arg(z_1 z_2) = \arg z_1 + \arg z_2$**

Note You multiply the moduli and add the arguments.

To prove these results, consider z_1 and z_2 in modulus–argument form:

$z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$

Multiplying these numbers together, you get

$$\begin{aligned}
z_1 z_2 &= r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \times r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)\\
&= r_1 r_2(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 + \mathrm{i}\sin\theta_2)\\
&= r_1 r_2(\cos\theta_1\cos\theta_2 + \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 + \mathrm{i}^2\sin\theta_1\sin\theta_2)\\
&= r_1 r_2(\cos\theta_1\cos\theta_2 + \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\\
&= r_1 r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))\\
&= r_1 r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))
\end{aligned}$$

Links The last step of this working makes use of the trigonometric addition formulae:
$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$
$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$

This complex number is in modulus–argument form, with modulus $r_1 r_2$ and argument $\theta_1 + \theta_2$, as required.

You can derive similar results for dividing two complex numbers given in modulus–argument form.

- **For any two complex numbers z_1 and z_2,**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - **$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$**

Note You divide the moduli and subtract the arguments.

To prove these results, again consider z_1 and z_2 in modulus–argument form:

$z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$

Dividing z_1 by z_2 you get

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)}\\
&= \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)} \times \frac{(\cos\theta_2 - \mathrm{i}\sin\theta_2)}{(\cos\theta_2 - \mathrm{i}\sin\theta_2)}\\
&= \frac{r_1(\cos\theta_1\cos\theta_2 - \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 - \mathrm{i}^2\sin\theta_1\sin\theta_2)}{r_2(\cos\theta_2\cos\theta_2 - \mathrm{i}\cos\theta_2\sin\theta_2 + \mathrm{i}\sin\theta_2\cos\theta_2 - \mathrm{i}^2\sin\theta_2\sin\theta_2)}\\
&= \frac{r_1((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}\\
&= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2))
\end{aligned}$$

Links The last step of this working makes use of the trigonometric addition formulae together with the identity $\sin^2\theta + \cos^2\theta \equiv 1$

This complex number is in modulus–argument form, with modulus $\frac{r_1}{r_2}$ and argument $\theta_1 - \theta_2$, as required.

Example

$z_1 = 3\left(\cos\frac{5\pi}{12} + \mathrm{i}\sin\frac{5\pi}{12}\right)$ and $4\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

a Find: **i** $|z_1 z_2|$ **ii** $\arg(z_1 z_2)$

b Hence write $z_1 z_2$ in the form: **i** $r(\cos\theta + \mathrm{i}\sin\theta)$ **ii** $x + \mathrm{i}y$

Solution

Example

Express $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)}$ in the form $x + \mathrm{i}y$.

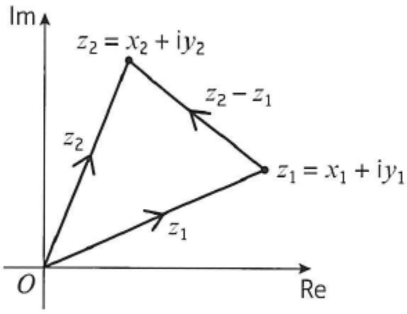
Solution

*Edexcel Book Ex2D Q1a,2,3ace,4ac

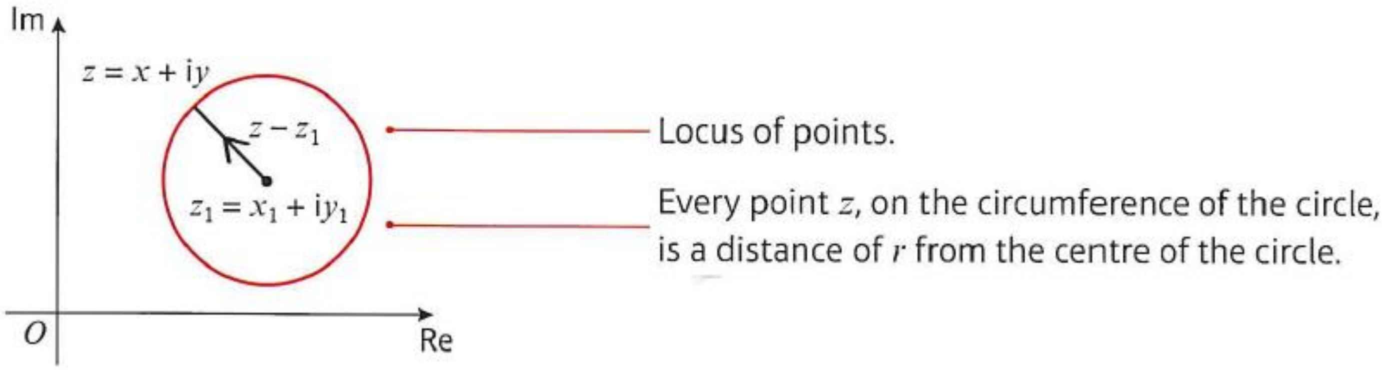
Loci in the Argand diagram

Complex numbers can be used to represent a locus of points on an Argand diagram.

- **For two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, $|z_2 - z_1|$ represents the distance between the points z_1 and z_2 on an Argand diagram.**



Using the above result, you can replace z_2 with the general point z. The locus of points described by $|z - z_1| = r$ is a circle with centre (x_1, y_1) and radius r.



- **Given $z_1 = x_1 + \mathrm{i}y_1$, the locus of point z on an Argand diagram such that $|z - z_1| = r$, or $|z - (x_1 + \mathrm{i}y_1)| = r$, is a circle with centre (x_1, y_1) and radius r.**

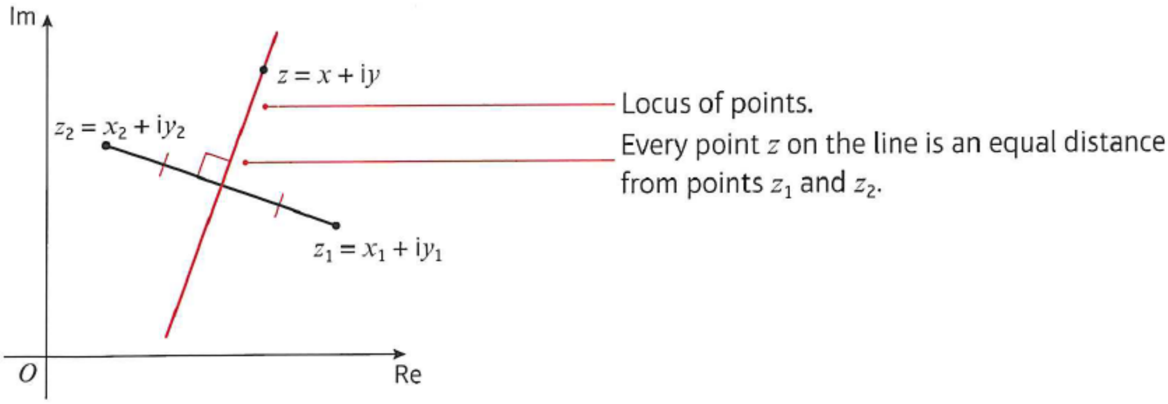
You can derive a Cartesian form of the equation of a circle from this form by squaring both sides:

$$|z - z_1| = r$$
$$|(x - x_1) + \mathrm{i}(y - y_1)| = r$$
$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

Since $|p + q\mathrm{i}| = \sqrt{p^2 + q^2}$

Links The Cartesian equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

The locus of points that are an equal distance from two different points z_1 and z_2 is the perpendicular bisector of the line segment joining the two points.



- **Given $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, the locus of points z on an Argand diagram such that $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line segment joining z_1 and z_2.**

Example

Given that z satisfies $|z-4|=5$,

a sketch the locus of z on an Argand diagram.

b Find the values of z that satisfy:

i both $|z-4|=5$ and $\text{Im}(z)=0$ **ii** both $|z-4|=5$ and $\text{Re}(z)=0$

Solution

Example

A complex number z is represented by the point P in the Argand diagram.

Given that $|z-5-3\text{i}|=3$,

a sketch the locus of P **b** find the Cartesian equation of this locus

c find the maximum value of $\arg z$ in the interval $(-\pi, \pi)$.

Solution

Example

Given that the complex number $z = x + iy$ satisfies the equation $|z - 12 - 5i| = 3$, find the minimum value of $|z|$ and maximum value of $|z|$.

Solution

Example

Given that $|z - 3| = |z + i|$,

a sketch the locus of z and find the Cartesian equation of this locus

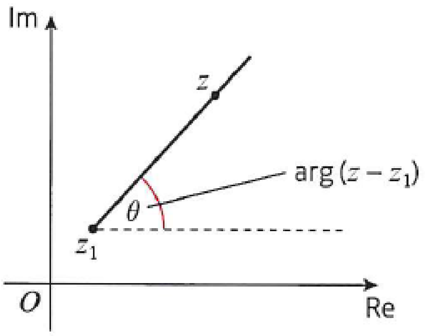
b find the least possible value of $|z|$.

Solution

Locus questions can also make use of the geometric property of the argument.

- **Given $z_1 = x_1 + \mathrm{i}y_1$, the locus of points z on an Argand diagram such that $\arg(z - z_1) = \theta$ is a half-line from, but not including, the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.**

Notation A **half-line** is a straight line extending from a point infinitely in one direction only.



You can find the Cartesian equation of the half-line corresponding to $\arg(z - z_1) = \theta$ by considering how the argument is calculated:

$$\arg(z - z_1) = \theta$$

$$\arg((x - x_1) + \mathrm{i}(y - y_1)) = \theta$$

$$\frac{y - y_1}{x - x_1} = \tan\theta$$

θ is a fixed angle so $\tan\theta$ is a constant.

$$y - y_1 = \tan\theta(x - x_1)$$

This is the equation of a straight line with gradient $\tan\theta$ passing through the point (x_1, y_1).

Example

Given that $\arg(z + 3 + 2\mathrm{i}) = \frac{3\pi}{4}$,

a sketch the locus of z on an Argand diagram

b find the Cartesian equation of the locus

c find the complex number z that satisfies both $|z + 3 + 2\mathrm{i}| = 10$ and $\arg(z + 3 + 2\mathrm{i}) = \frac{3\pi}{4}$

Solution

*Edexcel Book Ex2E Q1acei,4,6acei,7,10ace

Regions in the Argand diagram

You can use complex numbers to represent regions on an Argand diagram.

Example

a On separate Argand diagrams, shade in the regions represented by:

i $|z - 4 - 2i| \leqslant 2$ **ii** $|z - 4| < |z - 6|$ **iii** $0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}$

b Hence, on the same Argand diagram, shade the region which satisfies

$\{z \in \mathbb{C} : |z - 4 - 2i| \leqslant 2\} \cap \{z \in \mathbb{C} : |z - 4| < |z - 6|\} \cap \left\{z \in \mathbb{C} : 0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}\right\}$

Solution

Example

Sketch the locus of z such that

$$|z-3| = 2|z-1+\mathrm{i}|$$

This means that the locus is the path of the point (x, y) such that the distance between $(3, 0)$ and (x, y) is twice the distance between $(1, -1)$ and (x, y). It is all very well to state this, but what sort of a curve does this give you? Since on this occasion the locus is not obvious, you need to resort to algebra. (You could have done this with the previous examples if you had got stuck.)

Example

Sketch the locus of z such that $\arg \frac{z-2}{z+5} = \frac{\pi}{4}$.

Solution

Result:

$\arg\left(\frac{z-a}{z-b}\right) = \lambda$ is an arc of a circle subtended from the points 'a' and 'b'.

*Edexcel Book Ex2F Q1aceg,2-6

* Edexcel Book Mixed Exercise 2 as extra

Summary of key points

1 You can represent complex numbers on an **Argand diagram**. The x-axis on an Argand diagram is called the **real axis** and the y-axis is called the **imaginary axis**. The complex number $z = x + iy$ is represented on the diagram by the point $P(x, y)$, where x and y are Cartesian coordinates.

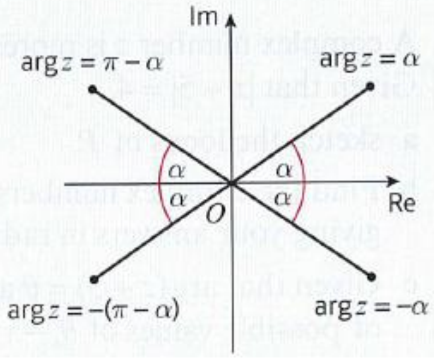
2 The complex number $z = x + iy$ can be represented as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on an Argand diagram.

3 The **modulus** of a complex number, $|z|$, is the distance from the origin to that number on an Argand diagram. For a complex number $z = x + iy$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$.

4 The **argument** of a complex number, $\arg z$, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram. For a complex number $z = x + iy$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$

5 Let α be the positive acute angle made with the real axis by the line joining the origin and z.

- If z lies in the first quadrant then $\arg z = \alpha$.
- If z lies in the second quadrant then $\arg z = \pi - \alpha$.
- If z lies in the third quadrant then $\arg z = -(\pi - \alpha)$.
- If z lies in the fourth quadrant then $\arg z = -\alpha$.



6 For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus–argument form of z is $z = r(\cos\theta + i\sin\theta)$

7 For any two complex numbers z_1 and z_2,

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

8 For two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, $|z_2 - z_1|$ represents the distance between the points z_1 and z_2 on an Argand diagram.

9 Given $z_1 = x_1 + iy_1$, the locus of points z on an Argand diagram such that $|z - z_1| = r$, or $|z - (x_1 + iy_1)| = r$, is a circle with centre (x_1, y_1) and radius r.

10 Given $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the locus of points z on an Argand diagram such that $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line segment joining z_1 and z_2.

11 Given $z_1 = x_1 + iy_1$, the locus of points z on an Argand diagram such that $\arg(z - z_1) = \theta$ is a half-line from, but not including, the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.