A21 Further Maths Hyperbolic Functions

The exponential functions can be combined to form functions that have strong similarities to trig (or circular) functions. These functions are called hyperbolic cosine (cosh x) and hyperbolic sine (sinh x).

$\cosh x = \frac{e^x+e^{-x}}{2}\ for\ x \in R$ similar to $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$\sinh x = \frac{e^x-e^{-x}}{2}\ for\ x \in R$ similar to $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

These two definitions are basic and from them four other hyperbolic functions are defined:-

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\therefore \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad for\ x \in R$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad for\ x \in R$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad for\ x \in R, x \neq 0$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^{2x} + 1}{e^{2x} - 1} \qquad for\ x \in R, x \neq 0$$

<u>Graphs of Hyperbolic Functions</u>

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = \frac{-(e^{x} - e^{-x})}{2} = -\sinh x$$

So $\sinh x$ is an odd function.

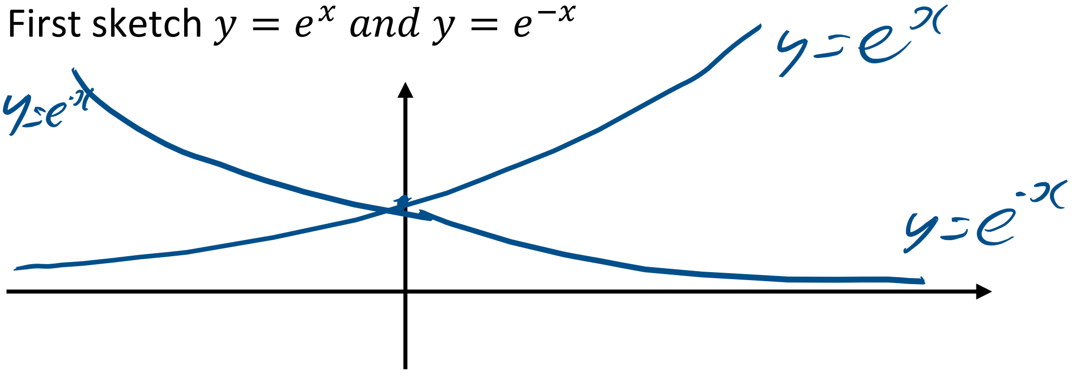
Similarly

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

So $\cosh x$ is an even function.

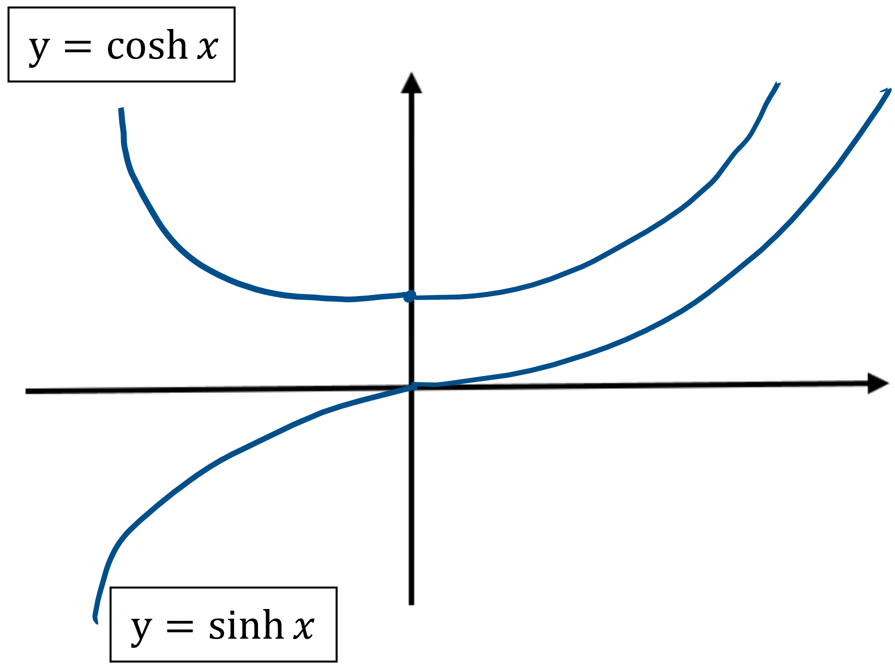
Also $\cosh x = \frac{e^{x}+e^{-x}}{2} > \frac{e^{x}-e^{-x}}{2} = \sinh x$ for all values.

First sketch $y = e^{x}$ and $y = e^{-x}$



$e^{x}>0$ and $e^{-x}>0$ for all x

So



Since $\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$, we see at $x = 0, \tanh x = 0$.

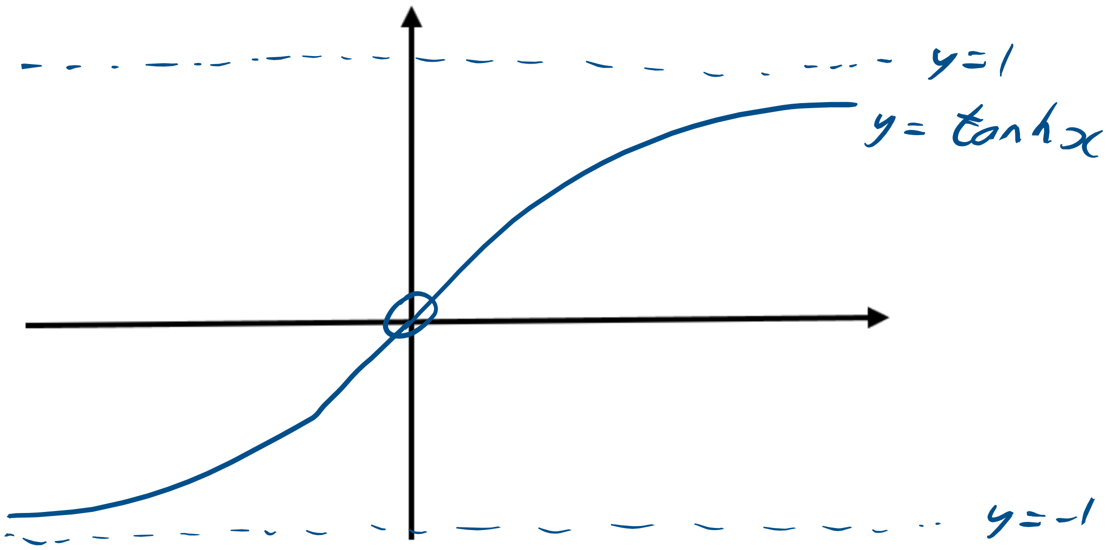
Also, $\tanh(-x) = \frac{e^{-2x}-1}{e^{-2x}+1} = \frac{\frac{1}{e^{2x}}-1}{\frac{1}{e^{2x}}+1} = \frac{1-e^{2x}}{1+e^{2x}} = -\tanh x$

So $\tanh x$ is an odd function.

Now $\tanh x = \frac{e^{2x}-1}{e^{2x}+1} = \frac{1-e^{-2x}}{1+e^{-2x}}$ (by dividing through by e^{2x})

As $x \to \infty, e^{-2x} \to 0$ and $\tanh x \to 1$

As $x \to -\infty, e^{2x} \to 0$ and $\tanh x \to -1$



The lines $y = \pm 1$ are asymptotes to the curve.

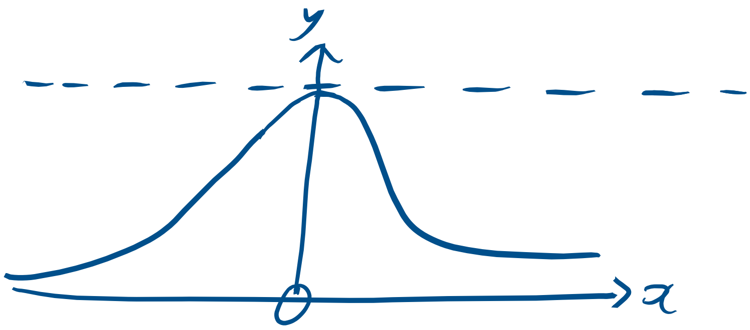
<u>Example</u> Sketch $y = \operatorname{sech} x$ $for\ x \in R$.

$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{1}{\cosh(-x)} = \operatorname{sech}(-x)$

$\therefore \operatorname{sech} x$ is an even function so symmetrical about y-axis

$\cosh x \geq 1 \Rightarrow \operatorname{sech} x$ lies in the interval $0 < \operatorname{sech} x \leq 1$.

$\operatorname{sech} 0 = \frac{1}{\cosh 0} = \frac{1}{1} = 1$



Example Find the exact values of x for which $\tanh x = \frac{1}{2}$.

$$\tanh x = \frac{e^{2x}-1}{e^{2x}+1} = \frac{1}{2}$$

$$\therefore \quad 2e^{2x} - 2 = e^{2x} + 1$$

$$e^{2x} = 3$$

$$2x = \ln 3$$

$$x = \tfrac{1}{2}\ln 3$$

Identities

Example Prove $cosh^2 x - sinh^2 x \equiv 1$

$$LHS = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4}$$

$$= \tfrac{1}{4}\left((e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})\right)$$

$$= \tfrac{1}{4}(4) = 1 \qquad \text{q.e.d.}$$

Note Since $\cosh^2 x - \sinh^2 x = 1$) $\div \cosh^2 x$

$\div \sinh^2 x$ $\quad 1 - \tanh^2 x = \operatorname{sech}^2 x$

$\coth^2 x - 1 = \operatorname{cosech}^2 x \qquad x \neq 0$

Example Prove $\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y$

$$RHS = \tfrac{1}{4}\left[(e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})\right]$$

$$= \tfrac{1}{4}\left[e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{-x+y} - e^{x-y} + e^{-x-y}\right]$$

$$= \tfrac{1}{4}\left[2e^{x+y} + 2e^{-(x+y)}\right]$$

$$= \tfrac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right]$$

$$= \cosh(x+y) \qquad \text{q.e.d.}$$

Note

By writing $x = y = A$ we get

$$\cosh 2A = \cosh^2 A + \sinh^2 A$$

Example Find an identity for $\sinh 2A$ in terms of $\cosh A$ and $\sinh A$.Hence find an identity for $\tanh 2A$.

$$\sinh 2A = \tfrac{1}{2}(e^{2A} - e^{-2A})$$
$$= \tfrac{1}{2}(e^{A} + e^{-A})(e^{A} - e^{-A})$$
$$= 2\left[\frac{e^{A}+e^{-A}}{2}\right]\left[\frac{e^{A}-e^{-A}}{2}\right]$$
$$= 2\cosh A \sinh A$$

hence $\sinh 2A = 2\sinh A \cosh A$

$\cosh 2A = \cosh^2 A + \sinh^2 A$ (from previous example)

$$\therefore \tanh 2A = \frac{2\sinh A \cosh A}{\cosh^2 A + \sinh^2 A}$$
$$= \frac{2\tanh A}{1+\tanh^2 A}$$

÷ through by $\cosh^2 A$

Osborne's Rule:- The formulae for circular and hyperbolic functions correspond exactly, provided the sign is changed whenever there exists a product (or implied product) of 2 sines.

i.e. the rule is to replace each trig function with its corresponding hyperbolic function and change the sign of every product (or implied product) of 2 sines.

e.g. $\cos 2A = 1 - sin^2 A$

becomes $\cosh 2A = 1 + sinh^2 A$

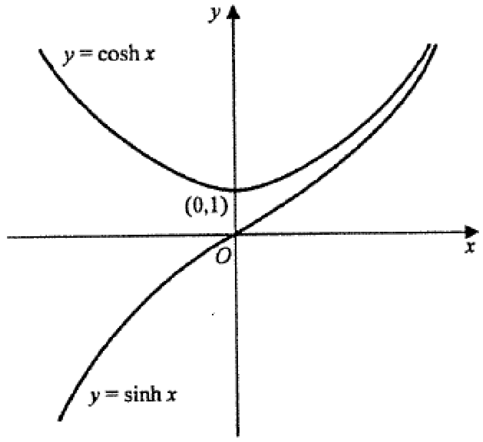
e.g. $\tan(A - B) = \frac{\tan A - \tan B}{1+\tan A \tan B}$

becomes $\tanh(A - B) = \frac{\tanh A - \tanh B}{1-\tanh A \tanh B}$

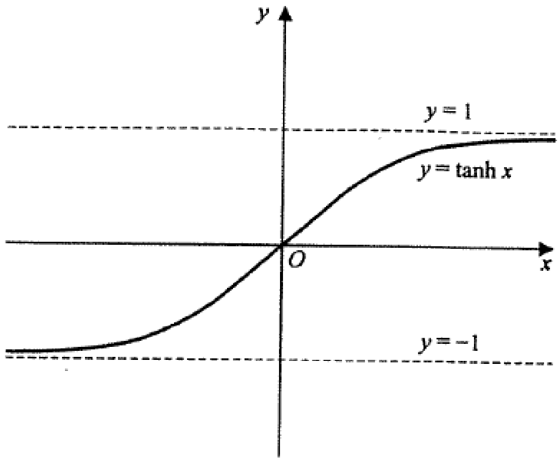
**P3 book Ex4A Q(1,2,3)alt parts, 4,5,7-17odds,18,20,22,23,25

Graphs of Hyperbolic Functions

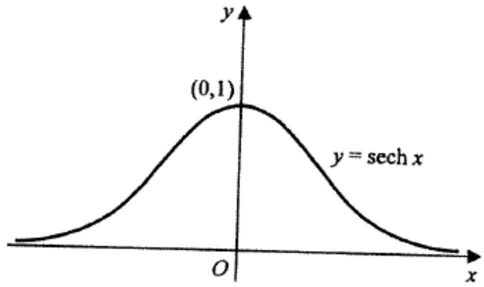
1.



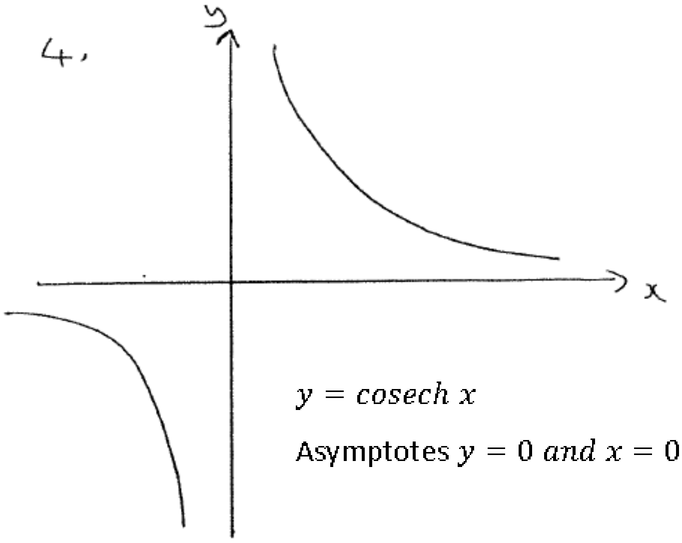
2.



3.



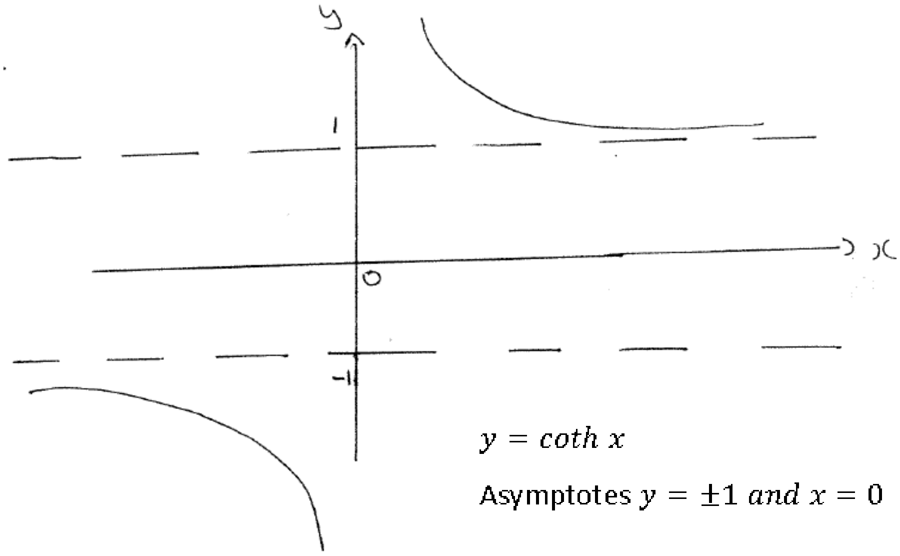
4.



$y = cosech\ x$

Asymptotes $y = 0\ and\ x = 0$

5.

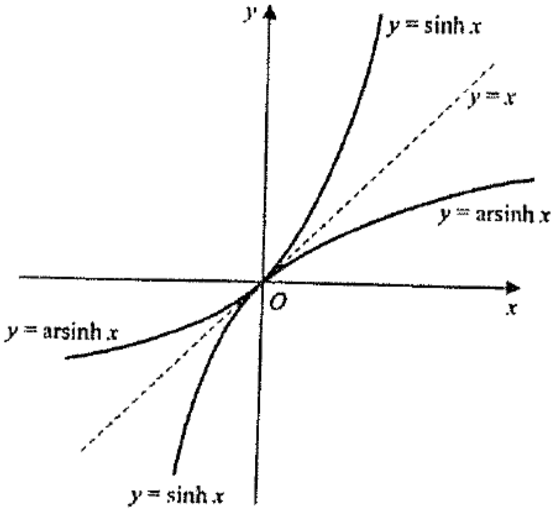


$y = coth\ x$

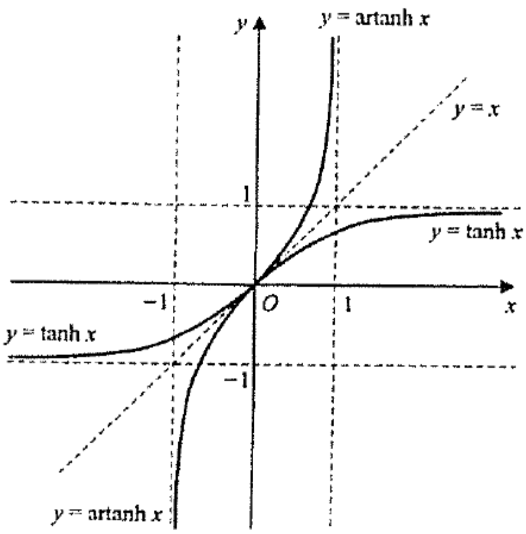
Asymptotes $y = \pm 1\ and\ x = 0$

Inverse Hyperbolic Functions

1.

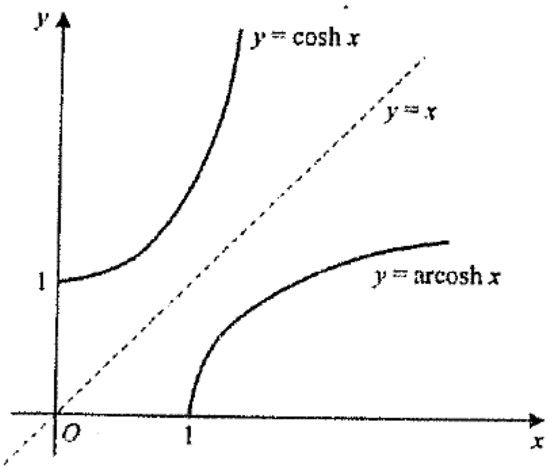


2.



3.

For the function $\cosh x$, you need to take the domain $x \geqslant 0$, so that it is a one–one function. Then the inverse function **arcosh x** is defined for the domain $x \geqslant 1$ and range $\operatorname{arcosh} x \geqslant 0$. The graphs of $\cosh x$ and $\operatorname{arcosh} x$ look like this:

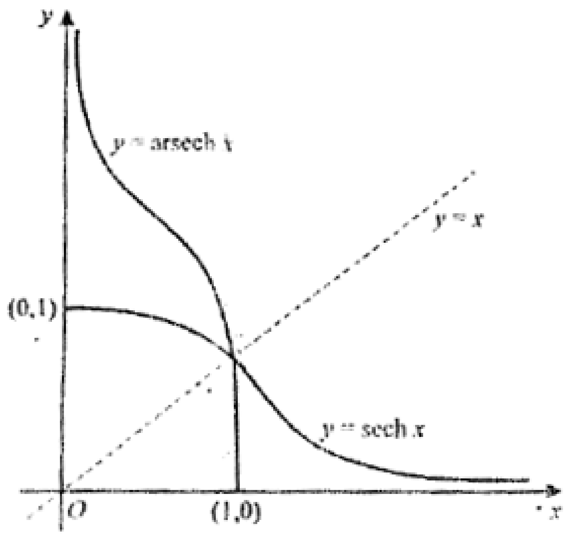


4.

In the same diagram, sketch the curves

$y = \operatorname{sech} x,\ x \in \mathbb{R},\ x \geqslant 0$
$y = \operatorname{arsech} x,\ x \in \mathbb{R},\ 0 < x \leqslant 1$

The curves are shown in the diagram. One is the reflection of the other in the line $y = x$.



The Logarithmic Form of Inverse Hyperbolic Functions

If $y = sinh^{-1}x \;\; then \; x = \sinh y$

Then $x = \frac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$2xe^y = e^{2y} - 1$

$0 = e^{2y} - 2xe^y - 1$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$e^y = x \pm \sqrt{x^2+1}$ but $e^y > 0$ *so take the positive root.*

$e^y = x + \sqrt{x^2+1}$

$y = \ln\left(x + \sqrt{x^2+1}\right)$

i.e. $sinh^{-1}x \;\; = \ln\left(x + \sqrt{x^2+1}\right)$

Similarly we can show:-

$cosh^{-1}x \;\; = \ln\left(x + \sqrt{x^2-1}\right)$ for $x \geq 1$

$tanh^{-1}x = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$

These results are given in the formula booklet

<u>Example</u> Express (a.) $\text{arcsinh}\,\frac{3}{4}$ (b.) $\text{arccosh}\,3$ (c.) $\text{arctanh}\,\frac{-3}{4}$ in log form.

(a) $\text{arcsinh}\left(\frac{3}{4}\right) = \ln\left[\frac{3}{4} + \sqrt{\frac{9}{16}+1}\right]$

$= \ln\left[\frac{3}{4} + \frac{5}{4}\right]$

$= \ln 2$

(b) $\text{arccosh}\,3 = \ln\left[3 + \sqrt{9-1}\right]$

$= \ln\left[3 + 2\sqrt{2}\right]$

(c) $\text{arctanh}\left(-\frac{3}{4}\right) = \frac{1}{2}\ln\left[\frac{1-\frac{3}{4}}{1+\frac{3}{4}}\right]$

$= \frac{1}{2}\ln\left(\frac{1}{7}\right) \;\; = -\frac{1}{2}\ln 7$

Example

Solve $sinh^2 x + 5 = 4\cosh x$

$* \cosh^2 x - \sinh^2 x = 1$

$\therefore \sinh^2 x = \cosh^2 x - 1$

$\therefore (\cosh^2 x - 1) + 5 = 4\cosh x$

$\cosh^2 x - 4\cosh x + 4 = 0$

$(\cosh x - 2)^2 = 0$

$\cosh x = 2$

$2 = \dfrac{e^x + e^{-x}}{2}$

$4 = e^x + e^{-x}$

$0 = e^{2x} - 4e^x + 1$

$e^x = \dfrac{4 \pm \sqrt{16 - 4}}{2}$

$e^x = 2 \pm \sqrt{3}$

$x = \ln(2 \pm \sqrt{3})$

* we cannot just use

$x = \cosh^{-1}(2)$

$= \ln(2 + \sqrt{2^2 - 1})$

$= \ln(2 + \sqrt{3})$

as this misses a root.

* Notice the equation

$\cosh x = 2$ has 2 roots,

but $\operatorname{arcosh} 2 = \ln(2 + \sqrt{3})$ only

**P3 book Ex4A Q26,27,29,31,32,33,35,38,40

The Derivatives of Hyperbolic Functions

$$\frac{d(\sinh x)}{dx} = \frac{d(\frac{e^x - e^{-x}}{2})}{dx}$$

$$\therefore \frac{d(\sinh x)}{dx} = \frac{1}{2}(e^x + e^{-x})$$

$$\therefore \frac{d(\sinh x)}{dx} = \cosh x$$

Also

$$\frac{d(\cosh x)}{dx} = \frac{d(\frac{e^x + e^{-x}}{2})}{dx}$$

$$\therefore \frac{d(\cosh x)}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$\therefore \frac{d(\cosh x)}{dx} = \sinh x$$

Example

Find $\frac{d(\tanh x)}{dx}$

$$\frac{d(\tanh x)}{dx} = \frac{d\left(\frac{\sinh x}{\cosh x}\right)}{dx}$$

$$= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}$$

$$= \frac{1}{\cosh^2 x}$$

$$= \operatorname{sech}^2 x$$

$$\therefore \frac{d(\tanh x)}{dx} = \operatorname{sech}^2 x$$

Example

Find $\frac{d(\coth x)}{dx}$

$$\frac{d(\coth x)}{dx}$$

$$= \frac{d\left(\frac{\cosh x}{\sinh x}\right)}{dx}$$

$$= \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x}$$

$$= \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x \qquad \therefore \frac{d(\coth x)}{dx} = -\operatorname{cosech}^2 x$$

Example

Find $\frac{d(\operatorname{sech} x)}{dx}$

$$= \frac{d(\cosh x)^{-1}}{dx}$$

$$= -1(\cosh x)^{-2} \sinh x$$

$$= \frac{-\sinh x}{\cosh^2 x} = -\tanh x \operatorname{sech} x$$

Example

Find $\frac{d(\operatorname{cosech} x)}{dx}$

$$= \frac{d(\sinh x)^{-1}}{dx} = -1(\sinh x)^{-2} \cosh x$$

$$= \frac{-\cosh x}{\sinh^2 x}$$

$$= -\coth x \operatorname{cosech} x$$

Example

Given $y = \cos x \cosh x$, $find\ \frac{d^2y}{dx^2}$.

$$y = \cos x \cosh x$$
$$\frac{dy}{dx} = \cos x \sinh x + \cosh x(-\sin x)$$
$$= \cos x \sinh x - \sin x \cosh x$$
$$\frac{d^2y}{dx^2} = (\cos x \cosh x - \sin x \sinh x) - (\sin x \sinh x + \cos x \cosh x)$$
$$= -2 \sin x \sinh x$$

Example

A curve is given by the equations $x = \cosh t$, $y = \sinh t$ where t is a parameter.

(a.) Find the cartesian equation of the curve.

(b.) Find the equation of the tangent at point where $t = \ln 2$.

(a) $x^2 = \cosh^2 t \quad y^2 = \sinh^2 t$

$\therefore x^2 - y^2 = 1$

(b) $\frac{dx}{dt} = \sinh t \quad \frac{dy}{dt} = \cosh t$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$
$$= \frac{\cosh t}{\sinh t}$$

when $t = \ln 2$ $\quad x = \frac{1}{2}[e^{\ln 2} + e^{-\ln 2}]$

$$x = \frac{1}{2}(2 + \tfrac{1}{2}) = \frac{5}{4}$$
$$y = \frac{1}{2}(e^{\ln 2} - e^{-\ln 2})$$
$$= \frac{1}{2}[2 - \tfrac{1}{2}] = \frac{3}{4}$$

so $\frac{dy}{dx} = \frac{\frac{5}{4}}{\frac{3}{4}} = \frac{5}{3}$

tangent is $y = mx + c$

$$y = \frac{5}{3}x + c$$

$(\frac{5}{4}, \frac{3}{4}) \quad \frac{3}{4} = \frac{5}{3}(\frac{5}{4}) + c$

$$\frac{3}{4} - \frac{25}{12} = c$$
$$c = -\frac{4}{3}$$

$\therefore y = \frac{5}{3}x - \frac{4}{3}$

or $3y = 5x - 4$

The Derivatives of Inverse Hyperbolic Functions

1. $y = \sinh^{-1} x$

$$x = \sinh y$$

$$\frac{dx}{dy} = \cosh y$$
$$\frac{dy}{dx} = \frac{1}{\cosh y}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{\sinh^2 y + 1}}$$

*take the positive sign as $\cosh y$ is positive for all y and use $'\cosh^2 y - \sinh^2 y = 1'$ to get...

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

$$\therefore \frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

2. $y = \cosh^{-1} x$

$$x = \cosh y$$

$$\frac{dx}{dy} = \sinh y$$
$$\frac{dy}{dx} = \frac{1}{\sinh y}$$
$$\frac{dy}{dx} = \frac{1}{\pm\sqrt{\cosh^2 y + 1}}$$
$$\frac{dy}{dx} = \frac{1}{\pm\sqrt{x^2 - 1}}$$

(but $\cosh^{-1} x$ is defined for $y \geq 0$ so $\sinh y \geq 0$)

$$\therefore \frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

3. $y = \tanh^{-1} x$

$$x = \tanh y$$

$$\frac{dx}{dy} = \operatorname{sech}^2 y$$
$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$$

Remember $1 - \operatorname{sech}^2 y = \tanh^2 y$ *so* $\frac{dy}{dx} = \frac{1}{1-\tanh^2 y}$

$$\frac{dy}{dx} = \frac{1}{1 - x^2}$$

$$\therefore \frac{d(\tanh^{-1} x)}{dx} = \frac{1}{1 - x^2}$$

4. $\frac{d\left(\sinh^{-1}\left(\frac{x}{a}\right)\right)}{dx} = \frac{1/a}{\sqrt{\left(\frac{x}{a}\right)^2 + 1}}$

$= \frac{1}{a\sqrt{\left(\frac{x}{a}\right)^2 + 1}}$

$= \frac{1}{\sqrt{x^2 + a^2}}$

5. $\frac{d\left(\cosh^{-1}\left(\frac{x}{a}\right)\right)}{dx} = \frac{1/a}{\sqrt{\left(\frac{x}{a}\right)^2 - 1}}$

$= \frac{1}{a\sqrt{\left(\frac{x}{a}\right)^2 - 1}}$

$= \frac{1}{\sqrt{x^2 - a^2}}$

6. $\frac{d\left(\tanh^{-1}\left(\frac{x}{a}\right)\right)}{dx} = \frac{\frac{1}{a}}{1 - \left(\frac{x}{a}\right)^2}$

$= \frac{\frac{1}{a}}{1 - \frac{x^2}{a^2}}$ $\times$ top & bottom by a^2

$= \frac{a}{a^2 - x^2}$

Results

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = sinh^{-1}\left(\frac{x}{a}\right) + c \quad or \quad \ln(x+\sqrt{x^2+a^2})$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = cosh^{-1}\left(\frac{x}{a}\right) + c \quad or \quad \ln(x-\sqrt{x^2+a^2}), \quad (x>a)$$

<u>Example</u> Find the equation of thhe tangent at the point where $x = \frac{-1}{2}$ to the curve with equation $y = \tanh^{-1} x$.

$y = \tanh^{-1} x$

$\frac{dy}{dx} = \frac{1}{1-x^2}$

when $x = -\frac{1}{2}$, $\frac{dy}{dx} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

when $x = -\frac{1}{2}$ $\quad y = \tanh^{-1}(-\frac{1}{2})$

$= \frac{1}{2}\ln\left(\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right)$

$= \frac{1}{2}\ln\frac{1}{3}$

$= -\frac{1}{2}\ln 3$

$*$ $y = \tanh^{-1} x$

$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ for $|x|<1$

(In formula booklet)

tangent $\quad y = \frac{4}{3}x + c$

$(-\frac{1}{2}, -\frac{1}{2}\ln 3)$ $\quad -\frac{1}{2}\ln 3 = \frac{4}{3}(-\frac{1}{2}) + c$

$-\frac{1}{2}\ln 3 + \frac{2}{3} = c$

$y = \frac{4}{3}x + \frac{2}{3} - \frac{1}{2}\ln 3$

*P3 book <u>Ex4B</u> Q1-19odds, 21-25,27-53odds,54,56-59