A21 Further Maths Reduction Formula

Integration by Parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Example Find $\int xe^x\,dx$.

$u = x \qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1 \qquad v = e^x$

$\therefore \int xe^x dx = xe^x - \int e^x dx$

$= xe^x - e^x + c$ ✓

Example Find $\int x\cos x\,dx$.

$u = x \qquad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1 \qquad v = \sin x$

$\therefore \int x\cos x\,dx = x\sin x - \int \sin x\,dx$

$= x\sin x - (-\cos x) + c$

$= x\sin x + \cos x + c$ ✓

Example Find $\int \ln x\,dx$.

rewrite as $\int 1.\ln x\,dx$

$u = \ln x \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x} \qquad v = x$

$\therefore \int 1.\ln x\,dx = x\ln x - \int x\left(\frac{1}{x}\right)dx$

$= x\ln x - \int 1\,dx$

$= x\ln x - x + c$ ✓

Example Find $\int x^2 e^{2x}\, dx$.

$u = x^2 \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 2x \quad v = \frac{1}{2}e^{2x}$

$\therefore \int x^2 e^{2x} dx = \frac{x^2}{2}e^{2x} - \int x e^{2x} dx \quad ..(1)$

consider $\int x e^{2x} dx$

let $u = x \quad \frac{dv}{dx} = e^{2x}$

$\frac{du}{dx} = 1 \quad v = \frac{1}{2}e^{2x}$

$\therefore \int x e^{2x} dx = \frac{x}{2}e^{2x} - \int \frac{1}{2} e^{2x} dx$

$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$

sub back into (1)

$$I = \frac{x^2}{2}e^{2x} - \frac{x}{2}e^{2x} + \frac{1}{4}e^{2x} + c$$

Example Find $I = \int e^x sinx\, dx$.

let $u = \sin x \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = \cos x \quad v = e^x$

$I = \int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx \quad ..(1)$

consider $\int e^x \cos x\, dx$

$u = \cos x \quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = -\sin x \quad v = e^x$

$\therefore \int e^x \cos x\, dx = e^x \cos x + \int e^x \sin x\, dx$

$$\therefore I = e^x \sin x - [e^x \cos x + I] + c$$

$$\therefore 2I = e^x \sin x - e^x \cos x + c$$

$$\therefore I = \frac{e^x}{2}(\sin x - \cos x) + c$$

<u>Example</u> Find $I = \int_0^1 x(x-1)^3\, dx$ (definite integral)

$u = x$ $\quad \frac{dv}{dx} = (x-1)^3$ $\quad \frac{d(\frac{1}{4}(x-1)^4)}{dx} = (x-1)^3$

$\frac{du}{dx} = 1$ $\quad v = \frac{1}{4}(x-1)^4$ $\quad \therefore \int (x-1)^3 dx = \frac{1}{4}(x-1)^4$

$$\therefore I = \left[\frac{x}{4}(x-1)^4\right]_0^1 - \int_0^1 \frac{1}{4}(x-1)^4 dx$$

$$I = 0 - \left[\frac{1}{20}(x-1)^5\right]_0^1$$

generally saves time to sub in limits here!

$$I = -\left[\frac{1}{20}(0)^5 - \frac{1}{20}(-1)^5\right]$$

$$I = -\left(\frac{1}{20}\right)$$

$$I = -\frac{1}{20}$$

*P2 Book Ex9D Q1,3,5,6,9,14,15,16,18,19

Reduction Formula

Example If $I_n = \int x^n e^{-x}\,dx$ evaluate I_3.

let $u = x^n \qquad \frac{dv}{dx} = e^{-x}$

$\frac{du}{dx} = nx^{n-1} \qquad v = -e^{-x}$

$$I_n = -x^n e^{-x} + \int n x^{n-1} e^{-x}\,dx$$

$$= -x^n e^{-x} + n\int x^{n-1} e^{-x}\,dx$$

$$\therefore I_n = -x^n e^{-x} + nI_{n-1} \qquad n \geq 1$$

*This relation is called a reduction formula

$$I_3 = -x^3 e^{-x} + 3I_2$$

$$I_2 = -x^2 e^{-x} + 2I_1$$

$$I_1 = -x e^{-x} + I_0$$

$$I_0 = \int x^0 e^{-x}\,dx$$

$$= \int e^{-x}\,dx$$

$$= -e^{-x} + c$$

$$\therefore I_1 = -xe^{-x} - e^{-x} + c$$

$$\therefore I_2 = -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) + c$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

$$\therefore I_3 = -x^3 e^{-x} + 3(-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}) + c$$

$$= -x^3 e^{-x} - 3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + c$$

$$= -e^{-x}(x^3 + 3x^2 + 6x + 6) + c$$

*notice we kept the constant at the end at each stage to avoid having to adjust it at each stage.

Example

If $I_n = \int_0^{\frac{\pi}{2}} sin^n x\, dx$, show that $I_n = \frac{n-1}{n} I_{n-2}\ for\ n \geq 2.$

Hence find (a.) I_5 and (b.) I_6

rewrite as $\int_0^{\pi/2} \sin^{n-1}x \sin x\, dx$

$u = \sin^{n-1}x \qquad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = (n-1)\sin^{n-2}x \cos x \qquad v = -\cos x$

$\therefore I_n = \left[-\cos x \sin^{n-1}x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}x \cos^2 x\, dx$

$I_n = 0 + (n-1)\int_0^{\pi/2} \sin^{n-2}x(1-\sin^2 x)\, dx$

$I_n = (n-1)\int_0^{\pi/2} \sin^{n-2}x - \sin^n x\, dx$

$I_n = (n-1)(I_{n-2} - I_n)$

$I_n = (n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = (n-1)I_{n-2}$

$nI_n = (n-1)I_{n-2}$

$I_n = \frac{n-1}{n} I_{n-2} \qquad n \geqslant 2$

(a) $I_5 = \frac{4}{5} I_3$

$I_3 = \frac{2}{3} I_1$

$I_1 = \int_0^{\pi/2} \sin x\, dx = \left[-\cos x\right]_0^{\pi/2}$

$= -\cos\frac{\pi}{2} + \cos 0$

$= -0 + 1$

$= 1$

$\therefore I_5 = \frac{4}{5} \times \frac{2}{3} \times 1$

$= \frac{8}{15}$

(b) $I_6 = \frac{5}{6} I_4$

$I_4 = \frac{3}{4} I_2$

$I_2 = \frac{1}{2} I_0$

$I_0 = \int_0^{\pi/2} 1\, dx = \frac{\pi}{2}$

$\therefore I_6 = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

$= \frac{5\pi}{32}$

<u>Example</u> Use the identity $sec^2 A \equiv 1 + tan^2 A$ to find a reduction formula for

$$I_n = \int_0^{\frac{\pi}{4}} tan^n x\, dx$$

Hence, evaluate (a.) $\int_0^{\frac{\pi}{4}} tan^5 x\, dx$ and (b.) $\int_0^{\frac{\pi}{4}} tan^6 x\, dx$

Writing the integral as $I_n = \int_0^{\pi/4} tan^{n-2} x \, tan^2 x \, dx$

and substituting $tan^2 x = sec^2 x - 1$, we get

$$I_n = \int_0^{\pi/4} tan^{n-2} x (sec^2 x - 1) dx$$

$$= \int_0^{\pi/4} tan^{n-2} x \, sec^2 x - tan^{n-2} x \, dx$$

but $\frac{d(\tan x)}{dx} = sec^2 x$

$$\therefore \frac{d(tan^{n-1} x)}{dx} = (n-1) \, tan^{n-2} x \, sec^2 x$$

$$\therefore \int tan^{n-2} x \, sec^2 x \, dx = \frac{1}{n-1} tan^{n-1} x$$

$$\therefore I_n = \left[\frac{1}{n-1} tan^{n-1} x\right]_0^{\pi/4} - I_{n-2}$$

$$\therefore I_n = \left[\frac{1}{n-1}(1)\right] - \left[\frac{1}{n-1}(0)\right] - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2} \quad \text{for } n \geq 2$$

(a.) $\int_0^{\pi/4} tan^5 x \, dx = I_5$

$$I_n = \frac{1}{n-1} - I_{n-2}$$

$$I_5 = \frac{1}{4} - I_3$$

$$I_3 = \frac{1}{2} - I_1$$

$$I_1 = \int_0^{\pi/4} \tan x \, dx = \left[\ln|\sec x|\right]_0^{\pi/4}$$
$$= \left[\ln\left|\frac{1}{\cos x}\right|\right]_0^{\pi/4}$$
$$= \ln\sqrt{2} - \ln 1$$
$$= \frac{1}{2}\ln 2$$

$$\therefore I_3 = \frac{1}{2} - \frac{1}{2}\ln 2$$
$$I_5 = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\ln 2\right)$$
$$= \frac{1}{2}\ln 2 - \frac{1}{4} \quad \checkmark$$

(b) $\int_0^{\pi/4} \tan^6 x \, dx = I_6$

$$I_6 = \frac{1}{5} - I_4$$
$$I_4 = \frac{1}{3} - I_2$$
$$I_2 = \frac{1}{1} - I_0$$
$$I_0 = \int_0^{\pi/4} 1 \, dx$$
$$= \pi/4$$

$$\therefore I_2 = 1 - \pi/4$$
$$I_4 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right)$$
$$= \frac{\pi}{4} - \frac{2}{3}$$
$$I_6 = \frac{1}{5} - \left(\frac{\pi}{4} - \frac{2}{3}\right)$$
$$= \frac{1}{5} + \frac{2}{3} - \frac{\pi}{4}$$
$$= \frac{13}{15} - \frac{\pi}{4} \quad \checkmark$$

*P4 book Ex5A Q1,2,3,5,7,9-13,15, extras Q4,6 (tricky)