**A21 Further Maths**

## Differentiation and Integration of Inverse Trig Functions

### Graphs of inverse trigonometric functions

$y = \arcsin x$

Remember we defined

$$y = \sin x$$

to have domain

$$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

and the range is

$$-1 \leqslant \sin x \leqslant 1$$

Our inverse

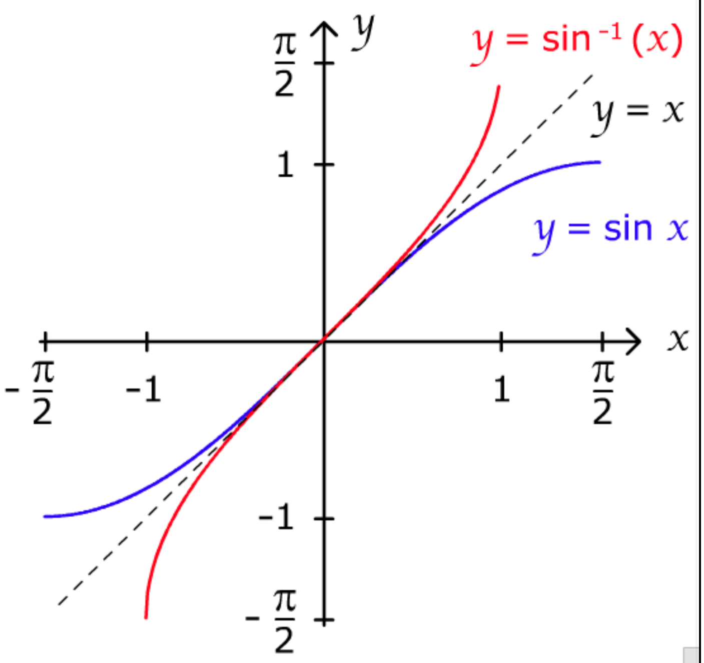
$$y = \sin^{-1} x$$

has domain

$$-1 \leqslant x \leqslant 1$$

and range

$$-\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$$




$y = \arccos x$

We limit the domain to

$$0 \leqslant x \leqslant \pi$$

and the range is

$$-1 \leqslant \cos x \leqslant 1$$

The inverse function looks like this.

It's a reflection of $y = \cos x$ in the line $y = x$.

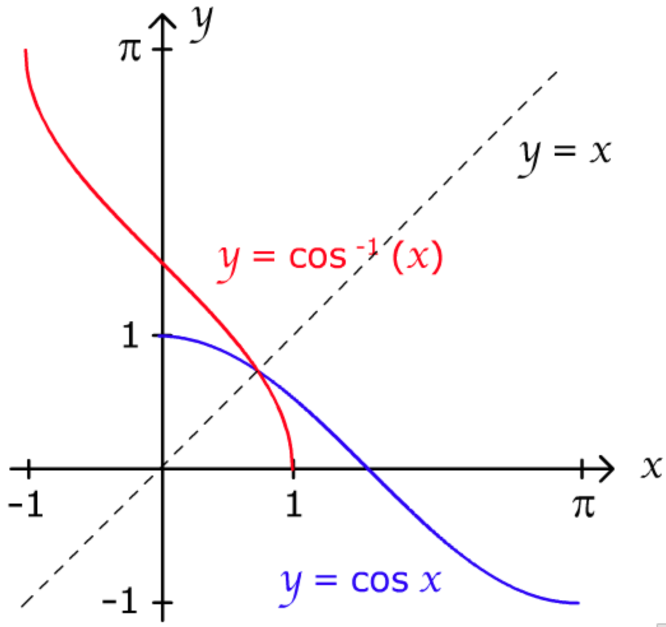
The domain of

$$y = \cos^{-1}(x)$$

is $\quad -1 \leqslant x \leqslant 1$

and the range is

$$0 \leqslant \cos^{-1}(x) \leqslant \pi$$



$\underline{y = \arctan x}$

Limit it to

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

Reflect it in the line $y = x$ ...

... to get the inverse $y = \tan^{-1}(x)$.

The asymptotes are also reflected.

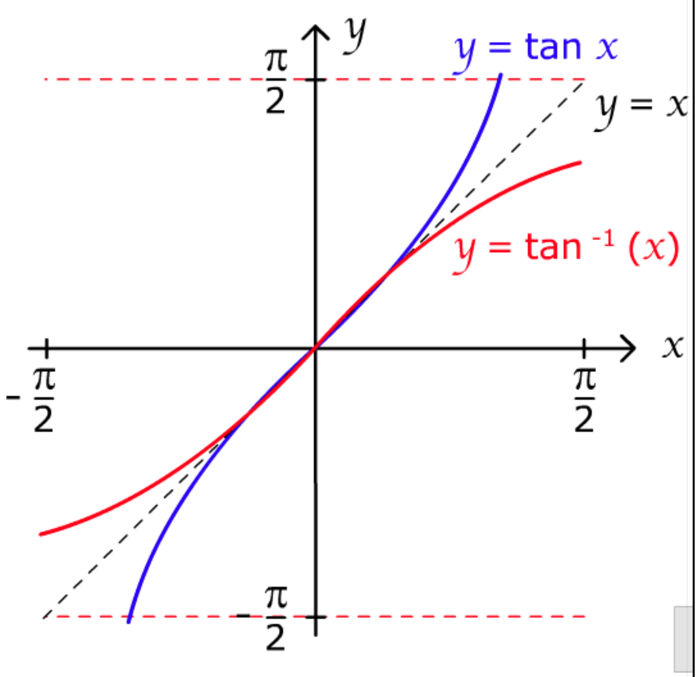
The domain of

$$y = \tan^{-1}(x)$$

is $x \in \mathbb{R}$

and the range is

$$-\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$$




## Differentiation of Inverse Trig Functions

1. Let $y = \sin^{-1} x \therefore x = \sin y \qquad -1 \leq x \leq 1$ and $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

$$\therefore \cos y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\pm\sqrt{1 - x^2}}$$

$* \, But \; y = \sin^{-1} x \; is \; an \; increasing \; function \; between$

$$-1 \; and \; 1, so \; \frac{dy}{dx} \; is \; positive.$$

$$\therefore \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

2. Let $\mathrm{y} = \cos^{-1} x \ \therefore x = \cos y \qquad -1 \le x \le 1$ and $0 \le y \le \pi$

$$\therefore\ -\sin y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\sin y}$$

$$\therefore \frac{dy}{dx} = \frac{-1}{\pm\sqrt{1-x^2}}$$

$* \mathit{But}\ \mathrm{y} = \cos^{-1} x\ \mathit{is\ a\ decreasing\ function\ between}$

$-1\ \mathit{and}\ 1, \mathit{so}\ \frac{dy}{dx}\ \mathit{is\ negative}.$

$$\therefore \frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

3. Let $\mathrm{y} = \tan^{-1} x \ \therefore x = \tan y\ -\infty < x < \infty$ (or $x \in R$)and $\frac{-\pi}{2} < y < \frac{\pi}{2}$

$$\therefore\ \sec^2 y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$* \sec^2 y = 1 + \tan^2 y$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\therefore \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

**Results**

$$\frac{d(sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}},$$

$$\frac{d(tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

Example Find $\frac{dy}{dx}$ when

(a.) $y = \cos^{-1} x^2$

Let $u = x^2$

$y = \cos^{-1} u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{-1}{\sqrt{1-u^2}} \times 2x$

$= \frac{-2x}{\sqrt{1-x^4}}$

(b.) $y = \tan^{-1}(e^{3x})$

Let $u = e^{3x}$

$y = \tan^{-1} u$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{1+u^2} \times 3e^{3x}$

$\frac{dy}{dx} = \frac{3e^{3x}}{1+e^{6x}}$

Example Find an equation of the normal to the curve $y = \sin^{-1} 2x$ at point where $x = \frac{1}{4}$.

$y = \sin^{-1}(2x)$

$\frac{dy}{dx} = \frac{2}{\sqrt{1-(2x)^2}}$

$= \frac{2}{\sqrt{1-4x^2}}$

when $x = \frac{1}{4}$

$\frac{dy}{dx} = \frac{2}{\sqrt{1-(\frac{1}{2})^2}} = \frac{2}{\sqrt{\frac{3}{4}}}$

$= \frac{4\sqrt{3}}{3}$

$\therefore$ grad normal $= -\frac{\sqrt{3}}{4}$ (negative reciprocal)

when $x = \frac{1}{4}$ $y = \sin^{-1}(2(\frac{1}{4}))$

$= \sin^{-1}(\frac{1}{2})$

$= \frac{\pi}{6}$

normal $y = -\frac{\sqrt{3}}{4}x + c$

$(\frac{1}{4}, \frac{\pi}{6})$ $\frac{\pi}{6} = -\frac{\sqrt{3}}{4}(\frac{1}{4}) + c$

$c = \frac{\pi}{6} + \frac{\sqrt{3}}{16}$

Normal is

$y = -\frac{\sqrt{3}}{4}x + \frac{\pi}{6} + \frac{\sqrt{3}}{16}$

*P3 Book Ex3C Q7-15

## ***Integration of*** $\frac{\mathbf{1}}{\boldsymbol{a^2+x^2}}$ ***and*** $\frac{\mathbf{1}}{\sqrt{\boldsymbol{a^2-x^2}}}$

1. Since $\frac{d(\sin^{-1}x)}{dx}=\frac{1}{\sqrt{1-x^2}}$

then $\frac{d(\sin^{-1}\frac{x}{a})}{dx}=\frac{1\times\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$

$\therefore \frac{d(\sin^{-1}\frac{x}{a})}{dx}=\frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}$ $\quad\therefore \frac{d(\sin^{-1}\frac{x}{a})}{dx}=\frac{1}{\sqrt{a^2-x^2}}$

$hence \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}+c$

2. And since $\frac{d(\tan^{-1}x)}{dx}=\frac{1}{1+x^2}$

then $\frac{d(\tan^{-1}\frac{x}{a})}{dx}=\frac{1\times\frac{1}{a}}{1+\left(\frac{x}{a}\right)^2}$

$\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx}=\frac{1}{a(1+\left(\frac{x}{a}\right)^2)}$

$\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx}=\frac{1}{a\left(\frac{a^2+x^2}{a^2}\right)}$

$\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx}=\frac{1}{a\left(\frac{a^2+x^2}{a^2}\right)}$

$\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx}=\frac{1}{\left(\frac{a^2+x^2}{a}\right)}$ $\quad\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx}=\frac{a}{a^2+x^2}$

$hence \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$

Example Find (a.) $\int \frac{4}{x^2+16}\,dx$ (b.) $\int \frac{2}{36+x^2}\,dx$

(a.) $\int \frac{4}{x^2+16}\,dx$

$= 4\int \frac{1}{x^2+(4)^2}\,dx$

$= 4 \times \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+c$

$= \tan^{-1}\left(\frac{x}{4}\right)+c$

(b.) $\int \frac{2}{36+x^2}\,dx$

$= 2\int \frac{1}{x^2+6^2}\,dx$

$= 2 \times \frac{1}{6}\tan^{-1}\left(\frac{x}{6}\right)+c$

$= \frac{1}{3}\tan^{-1}\left(\frac{x}{6}\right)+c$

Example Evaluate $\int_{-1.5}^{0} \frac{1}{\sqrt{9-x^2}}\,dx$

$\int_{-1.5}^{0} \frac{1}{\sqrt{9-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_{-1.5}^{0}$

$= \left[\sin^{-1}(0)\right] - \left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

$= [0] - \left[-\frac{\pi}{6}\right]$

$= \frac{\pi}{6}$

*UPM Ex15H Q13-24