What is an improper integral?

An improper integral is a definite integral for which the integrand (the expression to be integrated) is undefined either within at one or both of the limits of integration, or at some point between the limits of integration.

For example:

$\int_1^\infty \frac{1}{x^2}\,dx$ is an improper integral since one of its limits is infinity;

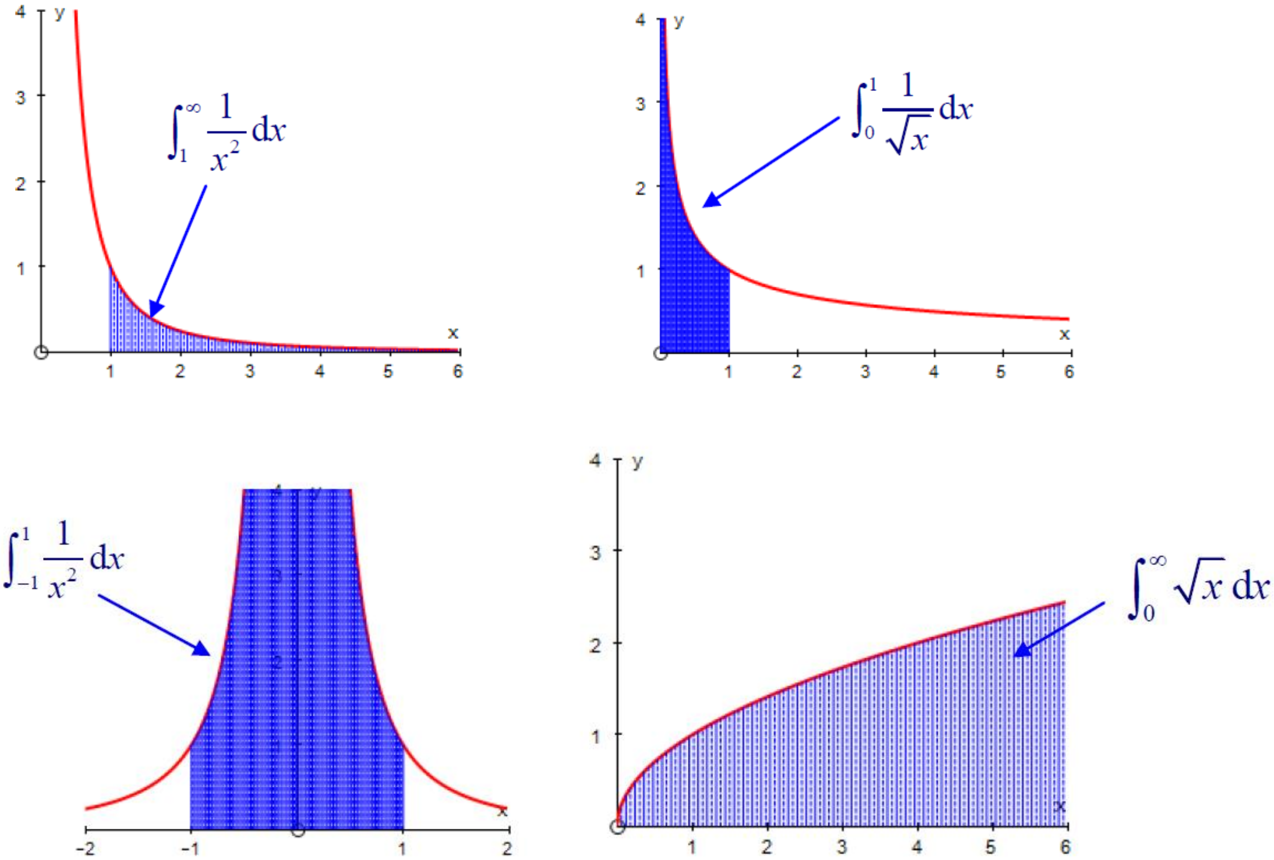
$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is an improper integral since it is undefined at $x = 0$.

$\int_{-1}^1 \frac{1}{x^2}\,dx$ is an improper integral since it is undefined at $x = 0$;

$\int_0^\infty \sqrt{x}\,dx$ is an improper integral since one of its limits is infinity;

Some improper integrals can be evaluated, others cannot. Remember that definite integration is equivalent to finding the area under a graph between two points. In some cases, an improper integral represents a finite area, even though the integrand is undefined at some point.

These are the graphs representing each of the four improper integrals above:



It is clear that the area in the fourth graph is infinite, and therefore that the value of the integral is undefined. However, in the cases of the other three graphs, the graphs are all approaching one of the axes, so it is possible that the area under the graphs may be finite, and therefore that the integral can be evaluated.

You can decide whether or not an improper integral has a finite value, and if so, what it is, by considering limits.

Improper integrals with limits involving infinity

For improper integrals with limits involving infinity, you replace the limit of infinity with a variable, work out the value of the integral in terms of the variable, and then look at what happens as the value of the variable tends to infinity.

Example 1

Find, if possible, the values of

(i) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

(ii) $\displaystyle\int_0^\infty \sqrt{x}\,dx$

Replace the limit ∞ by a

Solution

(i) $$\int_1^a \frac{1}{x^2}\,dx = \int_1^a x^{-2}\,dx = \left[-x^{-1}\right]_1^a = -\frac{1}{a}+1$$

As $a \to \infty$, $\frac{1}{a} \to 0$, so the value of the integral is 1.

Replace the limit ∞ by a

(ii) $$\int_0^a \sqrt{x}\,dx = \int_0^a x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^a = \tfrac{2}{3}a^{3/2}$$

As $a \to \infty$, $a^{3/2} \to \infty$, so the value of the integral cannot be found.

As expected from the graph, the integral $\int_0^\infty \sqrt{x}\,dx$ cannot be found.

However, the integral $\int_1^\infty \frac{1}{x^2}\,dx$ has been shown to have a finite value.

Improper integrals where the integral is undefined at a particular value

For improper integrals where the integrand is undefined at one of the limits of integration, you use a similar technique to the one above: you replace the limit with a variable, work out the value of the integral in terms of the variable, and then look at what happens as the value of the variable tends to the original value.

If the integrand is undefined at a point between the limits, you need to split the integral into two parts, so that the problem value is a limit of both parts, and then use the technique above.

Example 2

Find, if possible, the values of

(i) $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

(ii) $\int_{-1}^1 \frac{1}{x^2}\,dx$

> The integrand is not defined at $x = 0$, so replace the limit 0 by a

Solution

(i) $$\int_a^1 \frac{1}{\sqrt{x}}\,dx = \int_a^1 x^{-1/2}\,dx$$
$$= \left[2x^{1/2}\right]_a^1$$
$$= 2 - 2\sqrt{a}$$

As $a \to 0$, $\sqrt{a} \to 0$, so the value of the integral is 2.

(ii) $$\int_{-1}^1 \frac{1}{x^2}\,dx = \int_{-1}^a x^{-2}\,dx + \int_b^1 x^{-2}\,dx$$
$$= \left[-x^{-1}\right]_{-1}^a + \left[-x^{-1}\right]_b^1$$
$$= -\frac{1}{a} - 1 - 1 + \frac{1}{b}$$

> The integrand is not defined at $x = 0$, so split the integral into two parts and replace the limits 0 by a and b

As $a \to 0$, $\frac{1}{a}$ is not defined, and as $b \to 0$, $\frac{1}{b}$ is not defined.

So the value of the integral cannot be found.