A-Level Further Maths A21

## Maclaurins Series

Let f(x) be a function, which throughout a certain domain, including x=0 is

(a.) Differentiable any number of times ,and
(b.) The sum of a convergent power series.

Let this series be

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$so\ f(0) = a_0$$

*differentiating term by term and putting x=0

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots$$

$$so\ f'(0) = a_1$$

$$f''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots$$

$$so\ f''(0) = 2a_2$$

$$or\ f''(0) = 2!\,a_2$$

$$f'''(x) = 6a_3 + 24a_4 x + 60a_5 x^2 + \cdots$$

$$so\ f'''(0) = 6a_3$$

$$or\ f'''(0) = 3!\,a_2$$

$$so\ you\ could\ write\ f(x) = f(0) + xf'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \frac{x^4 f'^{v}(0)}{4!} + \cdots + \frac{x^n f^n(0)}{n!} + \cdots$$

This is Maclaurins Series.

Exponential Series

$Let\ f(x) = e^x$

$f(x) = e^x \quad \therefore f(0) = 1$

$f'(x) = e^x \quad \therefore f'(0) = 1$

$f(0) = f'(0) = f''(0) = \ldots = f^n(0) = 1$

$\therefore e^x = f(0) + x f'(0) + \frac{x^2 f''(0)}{2!} + \cdots \frac{x^n f^n(0)}{n!} + \ldots$

$= 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$

If $x = 1$

$e = 1 + 1 + \frac{1}{2} + \ldots + \frac{1}{n!} + \ldots$

$= 2.7182818$ (7d.p)

Logarithmic Series

Let $f(x) = \ln(1+x)$

$f(x) = \ln(1+x) \quad \therefore f(0) = 0$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \quad \therefore f'(0) = 1$

$f''(x) = -(1+x)^{-2} \quad \therefore f''(0) = -1$

$f'''(x) = 2(1+x)^{-3} \quad \therefore f'''(0) = 2 = 2!$

$f^{iv}(x) = -6(1+x)^{-4} \quad \therefore f^{iv}(0) = -6 = -3!$

$f^{v}(x) = 24(1+x)^{-5} \quad \therefore f^{v}(0) = 24 = 4!$

$\vdots \qquad \vdots$

$f^{n}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n} \quad \therefore f^{n}(0) = (-1)^{n-1}(n-1)!$

Maclaurin's gives:-

$$\ln(1+x) = x + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-3!) + \frac{x^5}{5!}(4!) + \ldots + \frac{x^n}{n!}(-1)^{n-1}(n-1)!$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots + \frac{(-1)^n x^n}{n} + \ldots \quad (1)$$

which has domain of convergence $-1 < x \leq 1$

Can't be $-1$ because $\ln 0$ is not defined.

Putting $-x$ for $x$, we obtain $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \ldots \quad (2)$

whose domain of convergence is $-1 < -x \leq 1$

ie. $1 > x \geq -1$

ie $-1 \leq x < 1$

(1) − (2) gives

$$\ln(1+x) - \ln(1-x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots$$

$$\therefore \ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\right) \ldots \quad (3)$$

which has domain of convergence $-1 < x < 1$ (ie. both (1) & (2) are convergent)

$\ln 2$ can be calculated using (3) by replacing $x$ by $\frac{1}{3}$

$$\ln 2 = 2\left(\frac{1}{3} + \frac{1}{81} + \frac{1}{5(3)^5} + \ldots\right) \approx 0.693$$ which converges more rapidly than (1).

Example

Expand $\cos x$ in ascending powers of x.

Solution

Let $f(x) = \cos x$ $\therefore f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$f^{iv}(x) = \cos x$ $\quad f^{iv}(0) = 1$

$$f(x) = f(0) + x f'(0) + \frac{x^2 f''(0)}{2!} + \frac{x^3 f'''(0)}{3!} + \dots + \frac{x^n f^n(0)}{n!} + \dots$$

$$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^r x^{2r}}{(2r)!}$$

valid for all values of $x$

***P3 Book Exercise 2D

(next bit is not needed. Just to show)

**Challenge**

The **ratio test** is a sufficient condition for the convergence of an infinite series. It says that a series $\sum_{r=1}^{\infty} a_r$ converges if $\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| < 1$, and diverges if $\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| > 1$.

Use the ratio test to show that

**a** the Maclaurin series expansion of $e^x$ converges for all $x \in \mathbb{R}$

**Problem-solving**

If $\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| = 1$ or does not exist then the ratio test is inconclusive.

$$e^x = \sum_{r=1}^{\infty} \frac{x^r}{r!}$$

$$= \lim_{r\to\infty}\left|\frac{\frac{x^{r+1}}{(r+1)!}}{\frac{x^r}{r!}}\right| = \lim_{r\to\infty}\left|\frac{x}{(r+1)}\right|$$ which is <1 for all x.

# Binomial Series

Consider $f(x) = (1+x)^n \quad for\ n \in R$

$$f(x) = (1+x)^n \quad so\ f(0) = 1$$

$$f'(x) = n(1+x)^{n-1} \quad so\ f'(0) = n$$

$$f''(x) = n(n-1)(1+x)^{n-2} \quad so\ f''(0) = n(n-1)$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \quad so\ f'''(0) = n(n-1)(n-2)$$

$$.$$

$$.$$

$$.$$

$$f^r(x) = n(n-1)(n-2)..(n-r+1)(1+x)^r \quad so\ f^r(0) = n(n-1)(n-2)..(n-r+1)$$

Maclaurins gives:-

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + .. + \frac{n(n-1)(n-2)..(n-r+1)x^r}{r!}$$

Which is the Binomial Series for any $n \in R$ and is convergent, provided $|x| < 1$.

If $n \in Z^+$, the series terminates and reduces to the Binomial Theorem.

<u>Note</u> Define $\binom{n}{r}$ to be

$$\binom{n}{r} = \frac{n(n-1)(n-2)..(n-r+1)}{r!}$$

<u>Example</u>

Expand $(1-3x)^{\frac{-2}{3}}$ up to terms including $x^3$.

<u>Solution</u>

$$(1-3x)^{-\frac{2}{3}} = 1 + \left(-\tfrac{2}{3}\right)(-3x) + \frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(-3x)^2}{2!} + \frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(-3x)^3}{3!} + \ldots$$

$$= 1 + 2x + 5x^2 + \frac{40}{3}x^3$$

valid for $|3x| < 1$

$|x| < \frac{1}{3}$

or $-\frac{1}{3} < x < \frac{1}{3}$

Example

Expand $(1-3x)^{\frac{1}{5}}$ inascending powers of x up to the term $x^3$. Take $x=\frac{1}{32}$ to find an approprimation for $29^{\frac{1}{5}}$, giving your answer correct to 5d.p.

Solution

$$(1-3x)^{\frac{1}{5}} = 1 + \left(\tfrac{1}{5}\right)(-3x) + \frac{\left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)(-3x)^2}{2!} + \frac{\left(\frac{1}{5}\right)\left(-\frac{4}{5}\right)\left(-\frac{9}{5}\right)(-3x)^3}{3!} + \ldots$$

$$= 1 - \frac{3}{5}x - \frac{18}{25}x^2 - \frac{162}{125}x^3 + \ldots$$

let $x=\frac{1}{32}$ $\therefore$ $\left(1-\frac{3}{32}\right)^{\frac{1}{5}} = 1 - \frac{3}{5}\left(\frac{1}{32}\right) - \frac{18}{25}\left(\frac{1}{32}\right)^2 - \frac{162}{125}\left(\frac{1}{32}\right)^3 + \ldots$

$\left(\frac{29}{32}\right)^{\frac{1}{5}} =$ " "

$\frac{1}{2}(29)^{\frac{1}{5}} =$ " "

$29^{\frac{1}{5}} = 2\times($ " " $)$

$= 1.96101$ (5dp)

Example

$$f(x) = \frac{x}{(3-2x)(2-x)}$$

(a.) Express $f(x)$ in partial fractions
(b.) Expand $f(x)$ up to terms including $x^3$.
(c.) State the set of values of x for which the series is valid.

Solution

(a) ans $= \frac{3}{3-2x} - \frac{2}{2-x}$ (working omitted)

(b) $f(x) = 3(3-2x)^{-1} - 2(2-x)^{-1}$

$= 3(3)^{-1}\left(1-\frac{2x}{3}\right)^{-1} - 2(2)^{-1}\left(1-\frac{x}{2}\right)^{-1}$

$= \left(1-\frac{2x}{3}\right)^{-1} - \left(1-\frac{x}{2}\right)^{-1}$

$= \left[1 + (-1)\left(-\frac{2x}{3}\right) + \frac{(-1)(-2)\left(\frac{2x}{3}\right)^2}{2!} + \frac{(-1)(-2)(-3)\left(-\frac{2x}{3}\right)^3}{3!}\right] - \left[1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-2)\left(-\frac{x}{2}\right)^2}{2!} + \frac{(-1)(-2)(-3)\left(-\frac{x}{2}\right)^3}{2!}\right]$

$= 1 + \frac{2x}{3} + \frac{4x^2}{9} + \frac{8x^3}{27} - 1 - \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{8}$

$= \frac{x}{6} + \frac{7x^2}{36} + \frac{37x^3}{216}$

(c) $\left(1-\frac{2x}{3}\right)^{-1}$ is valid for $\left|-\frac{2x}{3}\right|<1$ ie $|x|<\frac{3}{2}$

$\left(1-\frac{x}{2}\right)^{-1}$ is valid for $\left|-\frac{x}{2}\right|<1$ ie. $|x|<2$

so whole series valid for $|x|<\frac{3}{2}$

**Exercise 2C Q17,20-25

Using the Polynomial Series Form of Functions To Find Approximations For The Functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

So if you take terms in $x^3$ and higher powers of x to be negligible, then

$\sin x \approx x$ and $\cos x \approx 1 - \frac{x^2}{2}$ where x is small.

Also

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

So for small x, $\tan x \approx x$.

Example

Find a quadratic polynomial approximation for $\frac{\sin 2x}{1+x}$, give that x is small.

Solution

$\sin 2x \approx 2x$

$\frac{1}{(1+x)} = (1+x)^{-1} = 1 + (-1)(x) + \frac{(-1)(-2)(x)^2}{2!} + \frac{(-1)(-2)(-3)(x)^3}{3!} + \ldots$

$= 1 - x + x^2 - x^3 + \ldots$

So $\frac{\sin 2x}{(1+x)} = 2x(1 - x + x^2 - x^3 + \ldots)$

$\approx 2x - 2x^2$

powers of $x^3$ and higher being negligible as $x$ is small

Example

Given that x is small, show that $\frac{3\sin x}{2+\cos x} \approx x$.

Solution

$3\sin x \approx 3(x - \frac{x^3}{3!} + \ldots)$

$(2+\cos x)^{-1} = (2 + 1 - \frac{x^2}{2})^{-1}$

$= (3 - \frac{x^2}{2})^{-1}$

$= (3)^{-1}(1 - \frac{x^2}{6})^{-1}$

$= \frac{1}{3}(1 - \frac{x^2}{6})^{-1} = \frac{1}{3}(1 + (-1)(\frac{-x^2}{6}) + \ldots)$

$$\therefore \frac{3\sin x}{2+\cos x} \approx 3\left(x - \frac{x^3}{3!}\right)\frac{1}{3}\left(1+\frac{x^2}{6}\right)$$

$$\approx x + \frac{x^3}{6} - \frac{x^3}{6} - \frac{x^5}{36}$$

$$\approx x - \frac{x^5}{36}$$

$$\approx x$$

as $\frac{x^5}{36}$ can be neglected as $x$ is small

Example

Show that $\lim\limits_{x \to 0} \frac{1 - \cos 4x + x\sin 3x}{x^2} = 11$

Solution

$$\cos 4x \approx 1 - \frac{(4x)^2}{2}$$

$$= 1 - 8x^2$$

$$\sin 3x \approx 3x$$

so $$\frac{1 - \cos 4x + x\sin 3x}{x^2} = \frac{1 - (1 - 8x^2) + x(3x)}{x^2}$$

$$= \frac{8x^2 + 3x^2}{x^2}$$

$$= \frac{11x^2}{x^2}$$

$$= 11$$

Note other terms would have $x^2$, $x^4$ etc in them.

So $$\lim_{x \to 0} \frac{1 - \cos 4x + x\sin 3x}{x^2} = 11$$

**pg 29 Exercise 2E (use the standard Maclaurins results in the formula booklet)