A-level Further Maths A21

Induction

A theorem thought to be true for all values of the positive integer n, can be proved by showing that:-

(i) If it is true for n=k, then it is also true for n=k+1.

and

(ii) It is true for some small value of n such as n=1 (or perhaps n=2 or n=3)

If you prove both (i) and (ii) then you have shown that the theorem is true at the start (usually n=1) and it ids true for n=1+1 and n=2+1 and n=3+1 and so on for all integer values of n following on after the valid starting value (usually n=1).

Example 1 Use the method of mathematical induction to prove:-

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

where n is a positive integer.

Proof

Assume true for $n=k$

i.e. assume $1^3+2^3+3^3+\ldots+k^3 = \frac{1}{4}k^2(k+1)^2$

prove true for $n=k+1$

ie. prove $1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = \frac{1}{4}(k+1)^2(k+2)^2$

but $1^3+2^3+3^3+\ldots+k^3+(k+1)^3 = \frac{1}{4}k^2(k+1)^2+(k+1)^3$ by assumption

$\Rightarrow \sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k+1)^2[k^2+4(k+1)]$

$= \frac{1}{4}(k+1)^2(k^2+4k+4)$

$= \frac{1}{4}(k+1)^2(k+2)^2$ qed.

prove true for $n=1$

ie prove $\sum_{r=1}^{1} r^3 = \frac{1}{4}(1)^2(1+1)^2$

$1^3 = \frac{1}{4}(1)(4) = 1 \Rightarrow$ trivial

hence true for $n=1 \Rightarrow$ true for $n=2$ $\Rightarrow$ true for $n=3$ $\Rightarrow$ true for all positive integers.

Example 2 Use the method of mathematical induction to prove that the expression:-

$$3^{2n} + 7$$

Is divisible by 8 for all positive integers n.

Proof (Method 1)

Let $f(k) = 3^{2k} + 7$

then $f(k+1) = 3^{2(k+1)} + 7$

$= 3^{2k+2} + 7$

$\therefore f(k+1) - f(k) = 3^{2k+2} + 7 - 3^{2k} - 7$

$= 3^{2k+2} - 3^{2k}$

$= 3^{2k}(3^2 - 1)$

$= 3^{2k}(8)$

hence $f(k+1) = f(k) + 8(3^{2k})$

if we assume $f(k)$ is divis. by 8 and we can see that $8(3^{2k})$ is clearly divis. by 8 hence $f(k+1)$ which is the sum of those two terms, is divis. by 8.

Prove true for $n=1$ ie. prove 3^2+7 is divis. by 8.

$3^2 + 7 = 16$ which is divis. by 8.

hence true for $n=1 \Rightarrow$ true for $n=2 \Rightarrow$ true for all positive integers n.

**See other method too

Proof (Method 2)

$3^{2n}+7$

when $n=1$ $\quad 3^2+7=16$

which is divis. by 8.

Assume true for $n=k$

ie. $3^{2k}+7$ is divis. by 8

let $8Q=3^{2k}+7$ where Q is a whole number

prove true for $n=k+1$

$n=k+1$ $\quad 3^{2(k+1)}+7$

$=3^{2k+2}+7$

$=9(3^{2k})+7$ $\qquad * \; 8Q=3^{2k}+7$

$=9(8Q-7)+7$ $\qquad \therefore 3^{2k}=8Q-7$

$=9(8Q)-63+7$

$=9(8Q)-56$

$=8(9Q-7)$

So is divis. by 8

So true for $n=1 \Rightarrow$ true for $n=2 \Rightarrow$ true $\forall n \in \mathbb{Z}^+$

Example Given that n is an integer, which is greater than 3, show that

$$n! > 2^n$$

Proof

for $n=4$ $\quad n! = 4! = 24$

$2^n = 2^4 = 16$

So for $n=4$ $n! > 2^n$.

Assume true for some integer $k\,(>3)$ ie. $k! > 2^k$

Prove true for $n=k+1$

ie. prove $(k+1)! > 2^{k+1}$

proof :- $(k+1)! = (k!)(k+1)$

$> 2^k(k+1)$ by assumption

but $k>3$ $\therefore k+1>4$

and in particular $k+1>2$

$\therefore (k+1)! > 2^k . 2$

$\therefore (k+1)! > 2^{k+1}$

hence true for $n=4 \Rightarrow$ true for $n=5 \Rightarrow$ true for all positive integers $n>3$.

**P4 Book Page 279 Ex8A Q1-6,9,17,20,25,29,30,34, Extras: Q8,13,15