Summation of Finite Series Using The Method Of Differences

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n \ (frontwards)$$

$$\sum_{r=1}^{n} r = n + (n - 1) + (n - 2) \ldots 3 + 2 + 1 \ (backwards)$$

Adding:-

$$2\sum_{r=1}^{n} r = (n + 1) + (n + 1) + (n + 1) + \cdots (n + 1) + (n + 1) + (n + 1)$$

(n terms)

$$2\sum_{r=1}^{n} r = n(n + 1)$$

Result 1:-

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n + 1)$$

Note:- Here is another way you could sum the series $1 + 2 + 3 + \cdots + n$.

Consider the identity

$$2r \equiv r(r + 1) - (r - 1)r$$

Taking successive values 1,2,3,….,n for r, we get:-

$$2(1) = (1)(2) - (0)(1)$$
$$2(2) = (2)(3) - (1)(2)$$
$$2(3) = (3)(4) - (2)(3)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$2(n-1) = (n-1)(n) - (n-2)(n-1)$$
$$2n = n(n+1) - (n-1)(n)$$

adding up gives

$$2[1+2+3+\ldots+n] = n(n+1) - (0)(1)$$

$$\Rightarrow 2\sum_{r=1}^{n} r = n(n+1)$$

$$\Rightarrow \sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

This method is called summing a series by the method of difference.

Genereally if it is possible to find a function f® such that the rth term u_r of a series can be expressed as

$u_r = f(r+1) - f(r)$, then it is easy to find

$$\sum_{r=1}^{n} u_r$$

We have for r=1,2,3,...,n

$$u_1 = f(2) - f(1)$$

$$u_2 = f(3) - f(2)$$

$$u_3 = f(4) - f(3)$$

$$\because \quad \because \quad \because$$

$$\because \quad \because \quad \because$$

$$u_n = f(n+1) - f(n)$$

Adding:-

$$\sum_{r=1}^{n} u_r = f(n+1) - f(1)$$

because all the other terms on R.H.S. cancel out.

Example 1:-Find

$$\sum_{r=1}^{n} r^2$$

Consider the identity

$$24r^2 + 2 \equiv (2r+1)^3 - (2r-1)^3$$

And take r=1,2,3,...,n.

Solution

$$24(1)^2+2=(3)^3-(1)^3$$

$$24(2)^2+2=(5)^3-(3)^3$$

$$24(3)^2+2=(7)^3-(5)^3$$

$$\vdots \quad \vdots \quad \vdots$$

$$24(n-1)^2+2=(2n-1)^3-(2n-3)^3$$

$$24(n)^2+2=(2n+1)^3-(2n-1)^3$$

adding:-

$$24[1^2+2^2 \ldots n^2]+2n=(2n+1)^3-(1)^3$$

$$\Rightarrow 24\sum_{r=1}^{n} r^2 + 2n = 8n^3 + 12n^2 + 6n$$

$$24\sum_{r=1}^{n} r^2 = 8n^3 + 12n^2 + 4n$$

$$\sum_{r=1}^{n} r^2 = \frac{4n}{24}(2n^2+3n+1)$$

$$= \frac{n}{6}(2n+1)(n+1)$$

<u>Result 2</u>:-

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

<u>Example 2</u>:-Find

$$\sum_{r=1}^{n} r^3$$

Consider the identity

$$4r^3 \equiv r^2(r+1)^2 - (r-1)^2r^2$$

And take r=1,2,3,…,n.

<u>Solution</u>

$$4(1)^3 = (1)^2(2)^2 - (0)^2(1)^2$$

$$4(2)^3 = (2)^2(3)^2 - (1)^2(2)^2$$

$$4(3)^2 = (3)^2(4)^2 - (2)^2(3)^2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$4(n-1)^3 = (n-1)^2(n)^2 - (n-2)^2(n-1)^2$$

$$4(n)^3 = (n)^2(n+1)^2 - (n-1)^2(n)^2$$

adding

$$4(1^3+2^3+\dots n^3) = n^2(n+1)^2 - (0)^2(1)^2$$

$$4\sum_{r=1}^{n} r^3 = n^2(n+1)^2$$

$$\Rightarrow \sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Result 3:-

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Note:- Since

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Then

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2$$

Example 3:- Find

$$\sum_{r=1}^{n} r(r+1)$$

Consider the identity

$$3r(r+1) \equiv r(r+1)(r+2) - (r-1)(r)(r+1)$$

And take r=1,2,3,…,n.

Solution

$$3(1)(2) = (1)(2)(3) - (0)(1)(2)$$
$$3(2)(3) = (2)(3)(4) - (1)(2)(3)$$
$$3(3)(4) = (3)(4)(5) - (2)(3)(4)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$3(n-1)(n) = (n-1)(n)(n+1) - (n-2)(n-1)(n)$$
$$3(n)(n+1) = n(n+1)(n+2) - (n-1)(n)(n+1)$$

adding:-

$$3[(1)(2) + (2)(3) \ldots + (n)(n+1)] = n(n+1)(n+2)$$
$$3\sum_{r=1}^{n} r(r+1) = n(n+1)(n+2)$$

$$\Rightarrow \sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

Results for the sigma notation:-

1.

$$\sum_{r=1}^{n} af(r) = a\sum_{r=1}^{n} f(r)$$

Proof:-

$$\sum_{r=1}^{n} af(r) = af(1) + af(2) + af(3) + \cdots + af(n)$$

$$\sum_{r=1}^{n} af(r) = a[f(1) + f(2) + f(3) + \cdots + f(n)]$$

$$\therefore \sum_{r=1}^{n} af(r) = a\sum_{r=1}^{n} f(r)$$

2.

$$\sum_{r=1}^{n} f(r) + g(r) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$$

Proof:-

$$\sum_{r=1}^{n} f(r) + g(r) = f(1) + g(1) + f(2) + g(2) + \cdots f(n) + g(n)$$

$$\sum_{r=1}^{n} f(r) + g(r) = [f(1) + f(2) + \cdots + f(n)] + [g(1) + g(2) + \cdots + g(n)]$$

$$\sum_{r=1}^{n} f(r) + g(r) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$$

Questions: P3 book Page 15 Exercise 2A Q1,3,4,7,9,10

Telescoping Series

*Basically same as we have just done except to ∞.

Example:- Find the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

Solution

$$\frac{1}{(2n-1)(2n+1)} \equiv \frac{A}{2n-1} + \frac{B}{2n+1}$$

~~A n = 1/2 1 =~~ $\therefore \; 1 = A(2n+1) + B(2n-1)$

$n = \frac{1}{2} \qquad 1 = A(2) + B(0)$

$\Rightarrow A = \frac{1}{2}$

$n = -\frac{1}{2} \qquad 1 = -2B \qquad \Rightarrow B = -\frac{1}{2}$

hence $\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right]$

Let $u_r = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right]$

let $r = 1, 2, 3, \ldots n$

$u_1 = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right)$

$u_2 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$

$u_3 = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right]$

$\vdots$

$u_{n-1} = \frac{1}{2}\left(\frac{1}{2n-3} - \frac{1}{2n-1}\right]$

$u_n = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right]$

adding:-

$\sum_{r=1}^{n} u_r = \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]$

hence $\lim_{n\to\infty} \sum_{r=1}^{n} u_r = \frac{1}{2}(1-0]$

$= \frac{1}{2}$

hence

$\sum_{r=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}$

The above is an example of a telescoping series, since the terms of S_n, other than the first and last, cancel out in pairs.

Summation of Finite Series Using Standard Results

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Example:- Find

(a.) $\displaystyle\sum_{r=7}^{20} r^2$

(b.) $\displaystyle\sum_{r=12}^{25} r^3$

Solution

(a.) $\displaystyle\sum_{r=7}^{20} r^2 = \sum_{r=1}^{20} r^2 - \sum_{r=1}^{6} r^2$

$= \frac{1}{6}(20)(21)(41) - \frac{1}{6}(6)(7)(13)$

$= 2870 - 91$

$= 2779$

(b.) $\displaystyle\sum_{r=12}^{25} r^3 = \sum_{r=1}^{25} r^3 - \sum_{r=1}^{11} r^3$

$= \frac{1}{4}(25)^2(26)^2 - \frac{1}{4}(11)^2(12)^2$

$= 105625 - 4356$

$= 101269$

Example:- Show

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

Solution

$$\sum_{r=1}^{n} r(r+1) = \sum_{r=1}^{n} r^2 + r$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1)[(2n+1) + 3]$$

$$= \frac{1}{6}n(n+1)(2n+4)$$

$$= \frac{1}{3}n(n+1)(n+2)$$

q.e.d.

Example:- Find the following in terms of n.

$$\sum_{r=1}^{n} 6r^2 + 2^r$$

Solution

$$\sum_{r=1}^{n} 6r^2 + 2^r = \sum_{1}^{n} 6r^2 + \sum_{1}^{n} 2^r$$

$$= 6\sum_{1}^{n} r^2 + \sum_{1}^{n} 2^r \quad \ldots (1)$$

$$6\sum_{1}^{n} r^2 = 6 \times \frac{1}{6}n(n+1)(2n+1)$$

$$= n(n+1)(2n+1)$$

$$\sum_{1}^{n} 2^r = 2 + 2^2 + 2^3 + \ldots 2^n$$

*this is a GP with $a=2$, $r=2$, $n=n$

$$S_n = \frac{2(2^n-1)}{2-1}$$

hence (1) becomes

$$= n(n+1)(2n+1) + 2(2^n-1)$$

***Questions P3 book page 18 Ex2B Q1-3,5,-7,9-11,13,14,17,19