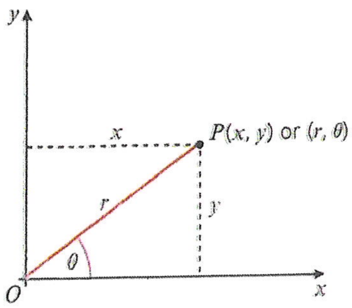
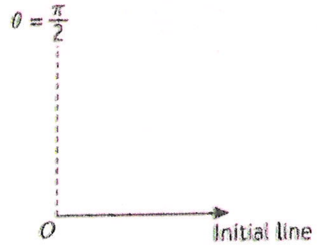
Teacher Copy

Polar Co-ordinates

Polar coordinates are an alternative way of describing the position of a point P in two-dimensional space. You need two measurements: firstly, the distance the point is from the **pole** (usually the origin O), r, and secondly, the angle measured anticlockwise from the **initial line** (usually the positive x-axis), θ. Polar coordinates are written as (r, θ).



Notation When working in polar coordinates the axes might also be labelled like this:



The coordinates of P can be written in either Cartesian form as (x, y) or in polar form as (r, θ).

You can convert between Cartesian coordinates and polar coordinates using right-angled triangle trigonometry.

From the diagram above you can see that:

- $r\cos\theta = x$
  $r\sin\theta = y$
- $r^2 = x^2 + y^2$
  $\theta = \arctan\left(\frac{y}{x}\right)$

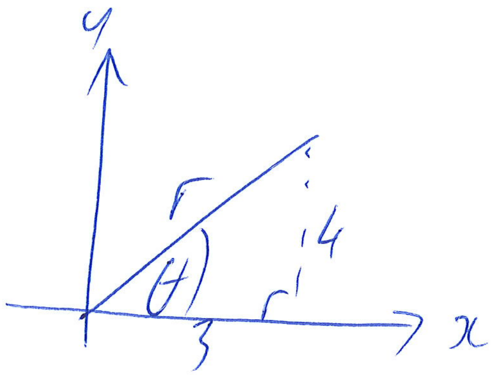
Watch out Always draw a sketch diagram to check in which quadrant the point lies, and always measure the polar angle from the positive x-axis.

Example

Find polar coordinates of the points with the following Cartesian coordinates.

a $(3, 4)$ **b** $(5, -12)$ **c** $(-\sqrt{3}, -1)$

(a)

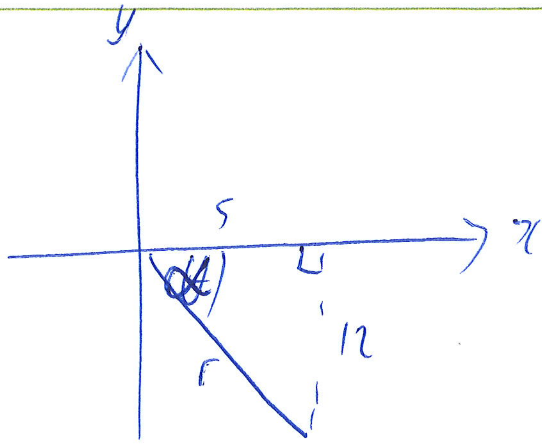


$r = \sqrt{3^2 + 4^2}$
$= 5$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$
$= 0.927...$

So polar coordinate is $(5, 0.927)$

(b)



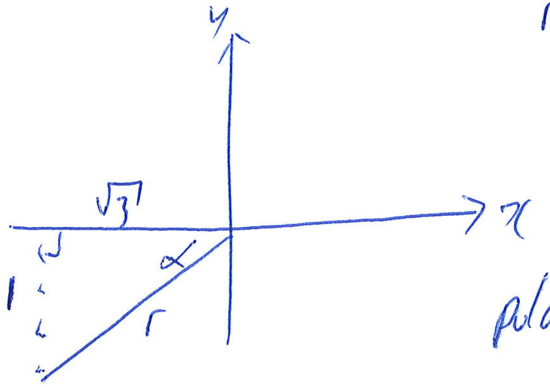
$r=\sqrt{(5)^2+(12)^2}=13$

$\alpha=\tan^{-1}\left(\frac{12}{5}\right)=1.176..$

so $\theta=-1.176$

polar co-ordinate is $(13, -1.176)$

(c)



$r=\sqrt{(\sqrt{3})^2+(1)^2}=2$

$\alpha=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$

so $\theta=\frac{7\pi}{6}$ or could say $\theta=-\frac{5\pi}{6}$

polar co-ordinate is $\left(2, \frac{7\pi}{6}\right)$

<u>Example</u>

Convert the following polar coordinates into Cartesian form. The angles are measured in radians.

a $\left(10, \frac{4\pi}{3}\right)$ **b** $\left(8, \frac{2\pi}{3}\right)$

(a) $x=r\cos\theta$

$=10\cos\frac{4\pi}{3}$

$=-5$

$y=r\sin\theta$

$=10\sin\frac{4\pi}{3}$

$=-5\sqrt{3}$

so $(-5, -5\sqrt{3})$

(b) $x=r\cos\theta$

$=8\cos\left(\frac{2\pi}{3}\right)$

$=-4$

$y=r\sin\left(\frac{2\pi}{3}\right)$

$=4\sqrt{3}$

so $(-4, 4\sqrt{3})$

Polar equations of curves are usually given in the form $r = f(\theta)$. For example,

$r = 2\cos\theta$

$r = 1 + 2\theta$

$r = 3$ —— In this example r is constant.

You can convert between polar equations of curves and their Cartesian forms.

Example

Find Cartesian equations of the following curves.

*remember $r^2=x^2+y^2$
$x=r\cos\theta,\ y=r\sin\theta$

a $r=5$ **b** $r=2+\cos 2\theta$ **c** $r^2=\sin 2\theta,\quad 0<\theta\leqslant\frac{\pi}{2}$

(a) $r=5$

square both sides

$r^2=5^2$

$x^2+y^2=25$ ✓

(b) $r=2+\cos 2\theta$

$r=1+(1+\cos 2\theta)$

$r=1+2\cos^2\theta$

$\times$ by r^2

$r^3=r^2+2r^2\cos^2\theta$

* $r\cos\theta=x$
* $r^2=x^2+y^2$

$(x^2+y^2)^{\frac{3}{2}}=(x^2+y^2)+2x^2$

$(x^2+y^2)^{\frac{3}{2}}=3x^2+y^2$ ✓

(c) $r^2=\sin 2\theta$

$r^2=2\sin\theta\cos\theta$) $\times r^2$

$r^4=2r\sin\theta\, r\cos\theta$

$(x^2+y^2)^2=2xy$ ✓

Example

Find polar equations for the following:

a $y^2=4x$ **b** $x^2-y^2=5$ **c** $y\sqrt{3}=x+4$

(a) $y^2=4x$

so $r^2\sin^2\theta=4r\cos\theta$

$r=\dfrac{4\cos\theta}{\sin^2\theta}$

$r=4\cot\theta\,\mathrm{cosec}\,\theta$ ✓

(b) $x^2-y^2=5$

$r^2\cos^2\theta-r^2\sin^2\theta=5$

$r^2(\cos^2\theta-\sin^2\theta)=5$

$r^2\cos 2\theta=5$

so $r^2=5\sec 2\theta$ ✓

(c) $y\sqrt{3}=x+4$

$r\sin\theta\times\sqrt{3}=r\cos\theta+4$

$r(\sqrt{3}\sin\theta-\cos\theta)=4$ ...①

let $\sqrt{3}\sin\theta-\cos\theta=R\sin(\theta-\alpha)$

$R\cos\alpha=\sqrt{3},\ R\sin\alpha=1$

$\tan\alpha=\frac{1}{\sqrt{3}}\ \therefore\ \alpha=\frac{\pi}{6}$

$R=\sqrt{(\sqrt{3})^2+(1)^2}=2$

① becomes $2r\sin(\theta-\frac{\pi}{6})=4$

so $r=2\,\mathrm{cosec}(\theta-\frac{\pi}{6})$ ✓

*Polar co-ordinate Question ExSA Q1-4

Sketching Curves

You can sketch curves given in polar form by learning the shapes of some standard curves.

- **$r = a$ is a circle with centre O and radius a.**
- **$\theta = \alpha$ is a half-line through O and making an angle α with the initial line.**
- **$r = a\theta$ is a spiral starting at O.**

Example

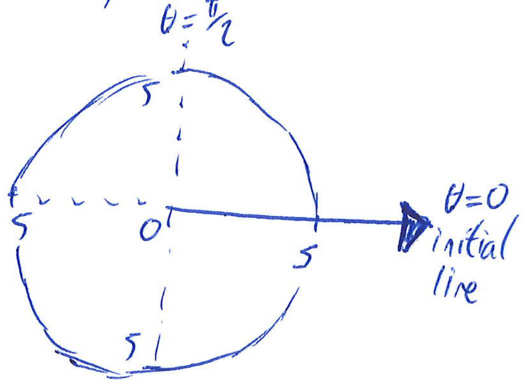
Sketch the following curves.

a $r = 5$ **b** $\theta = \frac{3\pi}{4}$ **c** $r = a\theta$

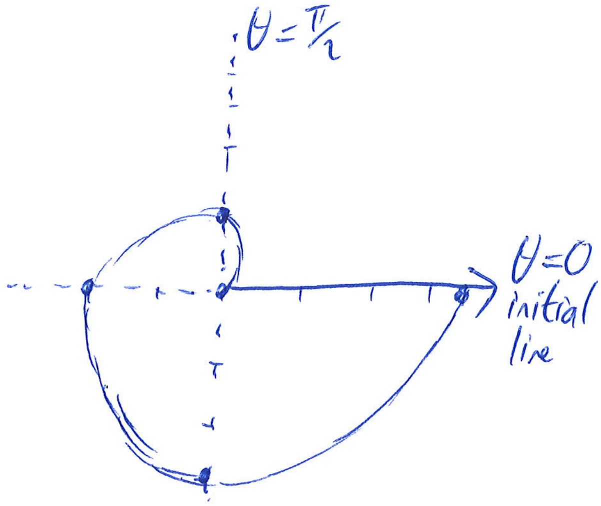
where a is a positive constant.

(a) $r=5$

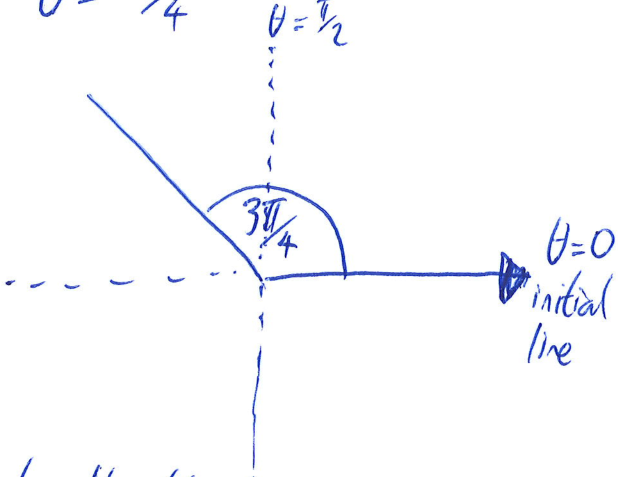
or $x^2+y^2=5$



(c) $r=a\theta$



(b) $\theta = 3\pi/4$



*note the other half of this line $y=-x$ would have equation $\theta = -\pi/4$ or $\theta = 7\pi/4$.

You can use a table of values to help.

*choose only values of θ that give positive values of r.

θ	0	$\frac{\pi}{2}$	π	$\frac{3}{2}\pi$	2π
r	0	$a\frac{\pi}{2}$	$a\pi$	$a\frac{3\pi}{2}$	$a2\pi$

Example

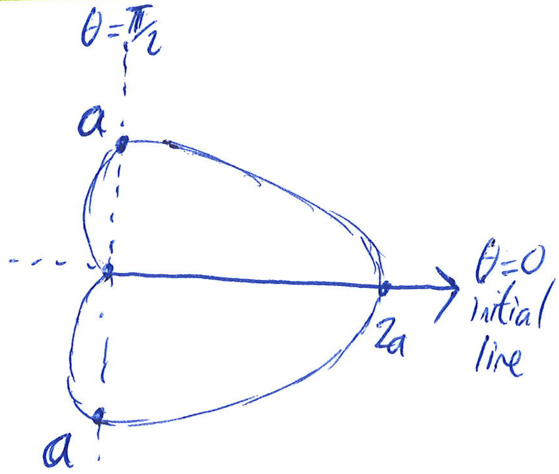
Sketch the following curves.

a $r = a(1 + \cos\theta)$ **b** $r = a\sin 3\theta$ **c** $r^2 = a^2\cos 2\theta$

(a) $r = a(1+\cos\theta)$

* plot at key pts where curve meets the axes.

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$2a$	a	0	a	$2a$



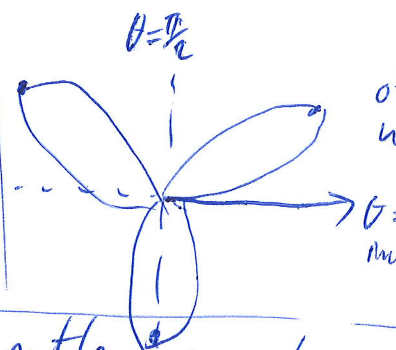
* the curve is 'heart' shaped and is known as a cardioid.

(b) $r = a\sin 3\theta$

Since we only draw when $r \geq 0$ you need to determine the values of θ required. Because $\sin\theta$ is positive for $0 \leq \theta \leq \pi$, $2\pi \leq \theta \leq 3\pi$, $4\pi \leq \theta \leq 5\pi$ then $\sin 3\theta$ is positive for $0 \leq \theta \leq \frac{\pi}{3}$, $\frac{2\pi}{3} \leq \theta \leq \pi$, $\frac{4\pi}{3} \leq \theta \leq \frac{5\pi}{3}$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$
r	0	a	0

for first loop, other 2 loops will have same values



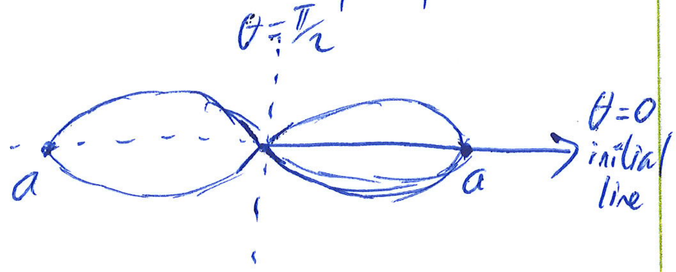
Note $r = a\sin 3\theta$ is typical of patterns in polar co-ordinates, $r = a\cos n\theta$ or $r = a\sin n\theta$ will have n loops symmetrically arranged around O.

(c) $r^2 = a^2\cos 2\theta$

* $\cos\theta$ is positive for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and again at $\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2}$

so $\cos 2\theta$ positive for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and $\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$

θ	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$
r	0	a	0	0	a	0



Curves with equations of the form $r = a(p + q\cos\theta)$ are defined for all values of θ if $p \geqslant q$. An example of this, when $p = q$, was the cardioid seen in Example 6a. These curves fall into two types, those that are 'egg' shaped (i.e. a convex curve) and those with a 'dimple' (i.e. the curve is concave at $\theta = \pi$). The conditions for each type are given below:

Links You can prove these conditions by considering the number of tangents to the curve that are perpendicular to the initial line.
→ Example 14

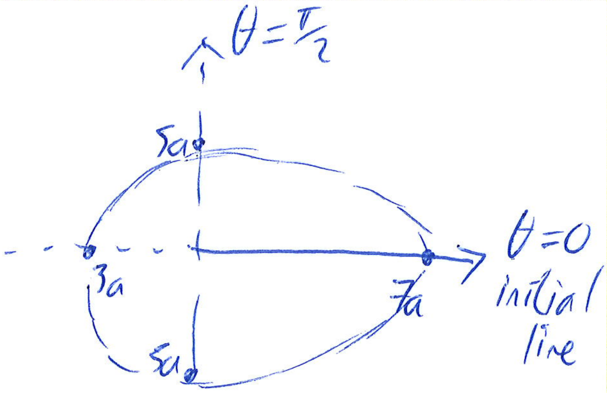
Example

Sketch the following curves.

a $r = a(5 + 2\cos\theta)$ **b** $r = a(3 + 2\cos\theta)$

(a) $r = a(5 + 2\cos\theta)$

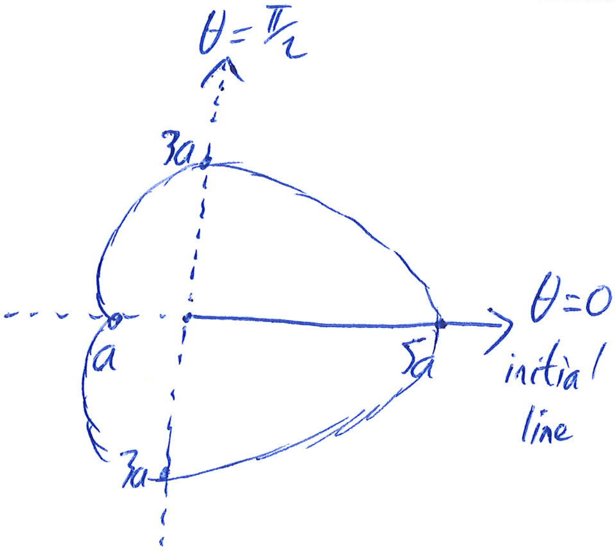
θ	0	$\pi/2$	π	$3\pi/2$
r	$7a$	$5a$	$3a$	$5a$



Since $5 > 2 \times 2$ there is no dimple.

(b) $r = a(3 + 2\cos\theta)$

θ	0	$\pi/2$	π	$3\pi/2$
r	$5a$	$3a$	a	$3a$



* Since $3 < 2 \times 2$ there will be a dimple for θ close to π

You may also need to find a polar curve to represent a locus of points on an Argand diagram.

Links If the pole is taken as the origin, and the initial line is taken as the positive real axis, then the point (r, θ) will represent the complex number $re^{i\theta}$
← Section 1.1

Example

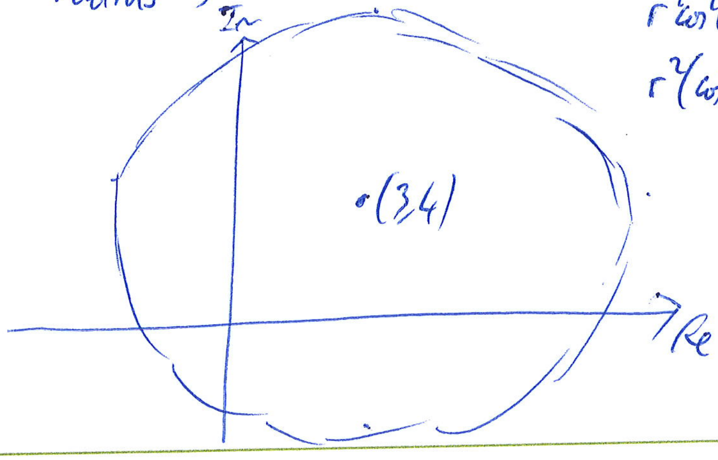
a Show on an Argand diagram the locus of points given by the values of z satisfying
$|z - 3 - 4i| = 5$

b Show that this locus of points can be represented by the polar curve $r = 6\cos\theta + 8\sin\theta$.

(a) $|z-(3+4i)|=5$

circle centre $(3,4)$

radius $= 5$



(b) cartesian form is

$(x-3)^2+(y-4)^2=25$

$(r\cos\theta-3)^2+(r\sin\theta-4)^2=25$

$r^2\cos^2\theta-6r\cos\theta+9+r^2\sin^2\theta-8r\sin\theta+16=25$

$r^2(\cos^2\theta+\sin^2\theta)-6r\cos\theta-8r\sin\theta=0$

$r^2=6r\cos\theta+8r\sin\theta$ ✓

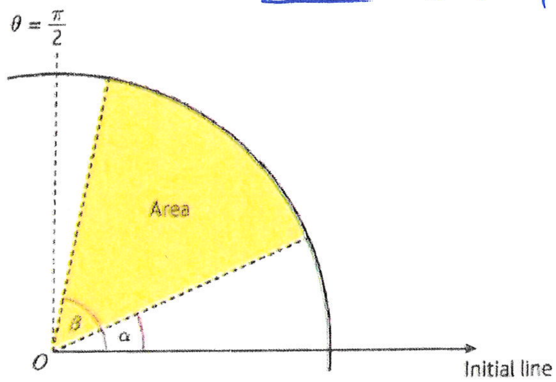
*Ex5B Q1-4

Area Enclosed By A Curve

You can find areas enclosed by a polar curve using integration.

- **The area of a sector bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by the formula**

  Area $= \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$

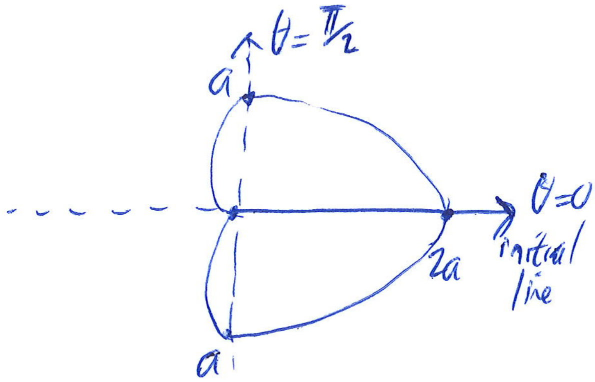


Example

Find the area enclosed by the cardioid with equation $r = a(1 + \cos\theta)$.

Sketch

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
r	$2a$	a	0	a	$2a$



* unlike cartesian integration areas in the 3rd & 4th quadrants do not produce negative areas so you just do

$$A = \frac{1}{2}\int_0^{2\pi} r^2\, d\theta$$

$$A = \frac{a^2}{2}\int_0^{2\pi} (1+\cos\theta)^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} 1 + 2\cos\theta + \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{2\pi} \frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\left[(3\pi) - (0)\right]$$

$$= \frac{3a^2\pi}{2}$$

Example

Find the area of one loop of the curve with polar equation $r = a\sin 4\theta$.

$r = a\sin 4\theta$ will have 4 loops in 2π. $r = a\sin 4\theta$ will have 1 loop for $0 \le \theta \le \frac{\pi}{4}$

* to find the beginning and end of a loop you solve $r = 0$

$0 = a\sin 4\theta$

$\sin 4\theta = 0$

$4\theta = 0$ or $4\theta = \pi$

$\theta = 0$ $\quad\theta = \frac{\pi}{4}$

$$\text{Area} = \frac{1}{2}\int_0^{\pi/4} r^2\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/4} \sin^2 4\theta\, d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi/4} (1 - \cos 8\theta)\, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{1}{8}\sin 8\theta\right]_0^{\pi/4}$$

$$= \frac{a^2}{4}\left[\frac{\pi}{4}\right] = \frac{a^2\pi}{16} \checkmark$$

Watch out $r = \sin n\theta$ has n loops and so a simple way of finding the area of one loop would appear to be to find $\frac{1}{2}\int_0^{2\pi} r^2\, d\theta$ and divide by n. This would give $\frac{a^2\pi}{8}$

The reason why this is not the correct answer is because when you take r^2 in the integral you are also including the n loops given by $r < 0$. You need to choose your limits carefully so that $r \geq 0$ for all values within the range of the integral.

Example

a On the same diagram, sketch the curves with equations $r = 2 + \cos\theta$ and $r = 5\cos\theta$.

b Find the polar coordinates of the points of intersection of these two curves.

c Find the exact area of the region which lies within both curves.

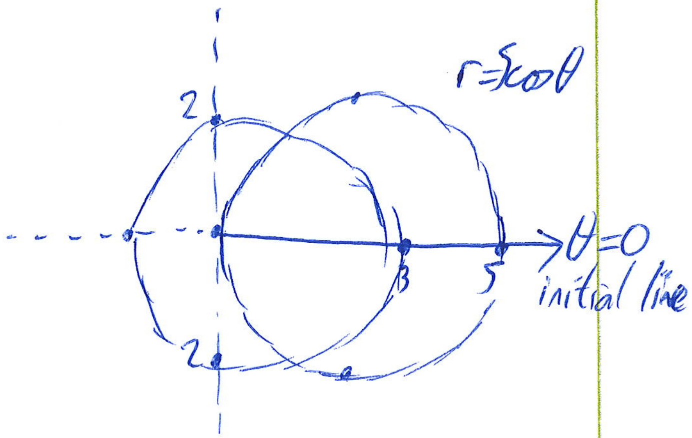
(a) $r = 2 + \cos\theta$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
r	3	2	1	2

$r = 5\cos\theta$

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{\pi}{4}$	$-\frac{\pi}{4}$	$-\frac{\pi}{2}$
r	5	0	-5	0	$\frac{5\sqrt{2}}{2}$	$\frac{5\sqrt{2}}{2}$	0

don't use as negative between $\pi/2$ & $3\pi/2$



(b) put $2 + \cos\theta = 5\cos\theta$

$$4\cos\theta = 2$$

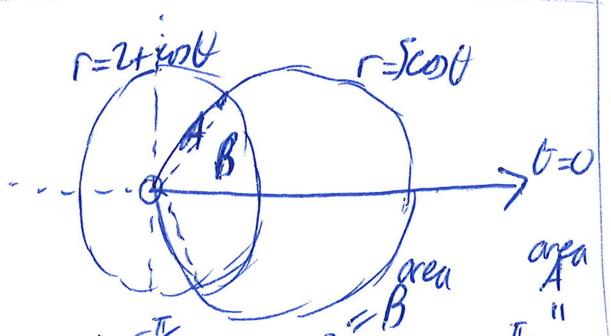
$$\cos\theta = \tfrac{1}{2}$$

$$\theta = \pm \tfrac{\pi}{3}$$

polar co-ordinates are $\left(\frac{5}{2}, \pm\frac{\pi}{3}\right)$

$r = 2 + \cos(\pi/3)$

(c)



$$\text{Area} = 2 \times \frac{1}{2}\int_0^{\pi/3} (2+\cos\theta)^2 \, d\theta + 2 \times \frac{1}{2}\int_{\pi/3}^{\pi/2} (5\cos\theta)^2 \, d\theta$$

$$= \int_0^{\pi/3} 4 + 4\cos\theta + \cos^2\theta \, d\theta + \int_{\pi/3}^{\pi/2} 25\cos^2\theta \, d\theta$$

(c) continued...

$$\text{Area} = \int_0^{\pi/3} \frac{9}{2} + 4\cos\theta + \frac{\cos 2\theta}{2} \, d\theta + \int_{\pi/3}^{\pi/2} \frac{25}{2}(1 + \cos 2\theta) \, d\theta$$

$$= \left[\frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/3} + \frac{25}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$

$$= \left[\frac{3\pi}{2} + 2\sqrt{3} + \frac{\sqrt{3}}{8}\right] + \frac{25}{2}\left[\left[\frac{\pi}{2} + 0\right] - \left[\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right]\right]$$

$$= \frac{43\pi}{12} - \sqrt{3}$$

*ESC

Tangents To Polar Curves

If you are given a curve $r = f(\theta)$ in polar form, you can write it as a parametric curve in Cartesian form, using θ as the parameter:

$$x = r\cos\theta = f(\theta)\cos\theta$$
$$y = r\sin\theta = f(\theta)\sin\theta$$

By differentiating parametrically, you can find the gradient of the curve at any point:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

When $\frac{dy}{d\theta} = 0$, a tangent to the curve will be horizontal.

When $\frac{dx}{d\theta} = 0$, a tangent to the curve will be vertical.

You need to be able to find tangents to a polar curve that are **parallel** or **perpendicular** to the initial line.

- **To find a tangent parallel to the initial line set $\frac{dy}{d\theta} = 0$.**
- **To find a tangent perpendicular to the initial line set $\frac{dx}{d\theta} = 0$.**

Example

Find the coordinates of the points on $r = a(1 + \cos\theta)$ where the tangents are parallel to the initial line $\theta = 0$.

$$y = r\sin\theta = a(1+\cos\theta)(\sin\theta)$$
$$= a(\sin\theta + \sin\theta\cos\theta)$$

$$\frac{dy}{d\theta} = a(\cos\theta + \cos^2\theta - \sin^2\theta)$$

put $\frac{dy}{d\theta} = 0 \quad \therefore \quad 0 = \cos\theta + \cos^2\theta - \sin^2\theta$

$$0 = \cos\theta + 2\cos^2\theta - 1$$
$$0 = 2\cos^2\theta + \cos\theta - 1$$
$$0 = (2\cos\theta - 1)(\cos\theta + 1)$$

$\cos\theta = \frac{1}{2} \qquad \cos\theta = -1$

$\therefore \theta = \pm\frac{\pi}{3} \qquad \theta = \pi$ and so $r = 0$

so $r = a(1 + \frac{1}{2}) = \frac{3a}{2}$

So tangents parallel to initial line are at $\left(\frac{3a}{2}, \pm\frac{\pi}{3}\right)$.

Note

You can see these tangents on a sketch of $y = a(1+\cos\theta)$

Example

Find the equations and the points of contact of the tangents to the curve $r = a\sin 2\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$ that are:

a parallel to the initial line

b perpendicular to the initial line.

Give answers to three significant figures where appropriate.

(a) $y = r\sin\theta$

$= a\sin\theta(\sin 2\theta)$

$\cancel{= 2a\sin^2\theta}$.

$\frac{dy}{d\theta} = a(\cos\theta\sin 2\theta + 2\sin\theta\cos 2\theta)$

$= a(2\cos\theta\sin\theta\cos\theta + 2\sin\theta\cos^2\theta - 2\sin^3\theta)$

$= 2a\sin\theta(\cos^2\theta + \cos^2\theta - \sin^2\theta)$

put $\frac{dy}{d\theta} = 0$ $\quad \sin\theta = 0$ $\quad$ or $\quad 2\cos^2\theta = \sin^2\theta$

$\therefore \theta = 0$ $\quad \therefore \tan^2\theta = 2$

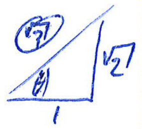
$\tan\theta = \pm\sqrt{2}$

$\theta = 0.955$

pick values in range of question

so pts are $(0,0)$

and if $\tan\theta = \sqrt{2}$



so $\sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$

and $\cos\theta = \frac{1}{\sqrt{3}}$

so when $\theta = 0.955$

$r = a \times 2 \times \sin\theta\cos\theta$

$= 2a \times \frac{\sqrt{2}}{\sqrt{3}} \times \frac{1}{\sqrt{3}}$

$r = \frac{2a\sqrt{2}}{3}$ ie. $\left(\frac{2a\sqrt{2}}{3}, 0.955\right)$

The equation of the initial line is $\theta = 0$ and that target is through $(0,0)$.

The equation of the target through $\left(\frac{2a\sqrt{2}}{3}, 0.955\right)$

is $y = \frac{2a\sqrt{2}}{3} \times \sin\theta$

$y = \frac{2a\sqrt{2}}{3} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{4a}{3\sqrt{3}}$

so equation of tangent is

$r = \frac{y}{\sin\theta}$

so $r = \frac{4a}{3\sqrt{3}}\operatorname{cosec}\theta$ ✓

(b) $x = r\cos\theta = a\cos\theta\sin 2\theta$

put $\frac{dx}{d\theta} = 0$ $\quad \frac{dx}{d\theta} = -a\sin\theta\sin 2\theta + 2a\cos\theta\cos 2\theta$

$0 = a(-2\sin^2\theta\cos\theta + 2\cos^3\theta - 2\cos\theta\sin^2\theta)$

$0 = 2a\cos\theta(-\sin^2\theta + \cos^2\theta - \sin^2\theta)$

$\cos\theta = 0$

so $\theta = \frac{\pi}{2}$

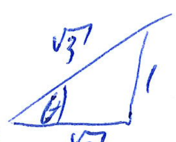
so y-axis is tangent

or $\cos^2\theta - 2\sin^2\theta = 0$

$\tan^2\theta = \frac{1}{2}$

$\tan\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = 0.615$



$\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$

$\sin\theta = \frac{1}{\sqrt{3}}$

so $r = 2a\sin\theta\cos\theta = 2a \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{2a\sqrt{2}}{3}$

tangent is at $\left(\frac{2a\sqrt{2}}{3}, 0.615\right)$

$x = \frac{2a\sqrt{2}}{3} \times \cos\theta = \frac{2a\sqrt{2}}{3} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{4a}{3\sqrt{3}}$ and equation of tangent is $r = \frac{x}{\cos\theta}$

so $r = \frac{4a}{3\sqrt{3}}\sec\theta$ ✓

*ESD

*Mixed Exercise 5

Summary of key points

1 For a point P with polar coordinates (r, θ) and Cartesian coordinates (x, y),

- $r\cos\theta = x$ and $r\sin\theta = y$
- $r^2 = x^2 + y^2$, $\theta = \arctan\left(\frac{y}{x}\right)$

Care must be taken to ensure that θ is in the correct quadrant.

2
- $r = a$ is a circle with centre O and radius a.
- $\theta = \alpha$ is a half-line through O and making an angle α with the initial line.
- $r = a\theta$ is a spiral starting at O.

3 The **area of a sector** bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by the formula

$$\textbf{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

4
- To find a tangent parallel to the initial line set $\frac{dy}{d\theta} = 0$.
- To find a tangent perpendicular to the initial line set $\frac{dx}{d\theta} = 0$.