Finding the general equation of a first order differential equation in which the variables are separable.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\therefore \frac{1}{g(y)} \frac{dy}{dx} = f(x)$$

$$\therefore \int \frac{1}{g(y)} dy = \int f(x) dx + c$$

Example

Given that y=2 at x=0 and $\frac{dy}{dx}=y^2+4$, find y in terms of x.

$$\frac{du}{dx} = y^{2} + 4$$

$$= 7 \int \frac{1}{y^{2} + 4} dy = \int dz$$

$$\frac{1}{z} \tan^{-1}(\frac{z}{z}) = 7 + ($$

$$\frac{1}{z} \tan^{-1}(\frac{z}{z}) = ($$

$$\frac{$$

^{*}P3 book <u>Ex8A</u> Q18-22

First Order Linear Differential Equaations

A 1st order linear differential equation is of the form

 $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x or constants.

An equation of this form is said to be $\underline{\text{exact}}$ when one side is the exact derivative of a product and the other side can be integrated wrt x.

If it is not exact then it can be made exact by multiplying through the equation by a function of *x*. This function is called the <u>integrating</u> <u>factor</u>.

Example

Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

Multiplying through by x gives...

$$x\frac{dy}{dx} + y = x^3$$

...making it exact.

$$\therefore \frac{d(xy)}{dx} = x^3$$

$$\therefore xy = \int x^3 dx$$

$$\therefore xy = \frac{x^4}{4} + c$$

In this case the integrating factor is x.

<u>Note</u>:- The integrating factor is given by f(x) where $f(x) = e^{\int P dx}$.

i.e. in the last example $P = \frac{1}{x}$

$$\therefore f(x) = e^{\int \frac{1}{x} dx}$$

$$\therefore f(x) = e^{\ln x}$$

$$\therefore f(x) = x$$

So the linear equation $\frac{dy}{dx} + Py = Q$ can be solved by multiplying by the integgrating factor $e^{\int Pdx}$, provided $e^{\int Pdx}$ can be found and the function $Qe^{\int Pdx}$ can be integrated wrt x.

Example

Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = \sin x \cos^3 x$$

$\cos x \frac{dy}{dx} + y \sin x = \sin x \cos^3 x$	
First divide though by cont to get in day + Py = Q. -> dy + sinh y = sinh cont 0	form
$ \frac{1}{1} = e^{-\int \frac{\sin dx}{\cos x}} $ $ = e^{-\int \frac{\sin dx}{\cos x}} $ $ = e^{-\int \frac{\sin x}{\cos x}} $ $ = e^{-\int \frac{\sin x}{\cos x}} $ $ = e^{-\int \frac{\sin x}{\cos x}} $	
mutt the 1 le ser 21	
=7 D being seix dy + tonx seix y = sin x cox => d(y seix)	
$\frac{1}{d2} = \sin x \cos x$	
$= \frac{dy \sec x}{dx} = \frac{1}{2} \sin 2x$	
4 Sec 2 = -4 colx +c	

Example

Find y in terms of x given that

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \ln x \text{ for } x > 0 \text{ and } y = 2 \text{ at } x = 1$$

$$IF = e^{\int_{-2}^{2} dx} = e^{-7\ln x} = e^{\ln x} = x^{-2}$$

$$= 7 \quad x^{-2} dx - 2x^{2} y = \ln x$$

$$= 7 \quad x^{-2} y = \int \ln x \, dx$$

$$= 7 \quad x^{-2} y = \int \ln x \, dx$$

$$= 1 \quad x + \int dx = \int dx = \int dx$$

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^{*}P3 book <u>Ex8C</u> Q1-9,13,14,16,17,18

The Second Order Linear Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
 where a, b and c are constants

The equation is called the 2nd order, because its highest derivative of y wrt x is $\frac{d^2y}{dx^2}$.

The equation is called linear because only 1st degree terms in y and its derivatives occur.

Result: The general solution of the 2nd order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is y = Au + Bv,

where y = u and y = v are particular, distinct solutions of the differential equation.

We now need to find the functions u and v in specific cases.

In the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, try as a solution...

 $y = e^{mx}$ where m is a constant to be found.

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

If $y = e^{mx}$ is a solution of the differential equation then

$$am^2e^{mx} + b me^{mx} + ce^{mx} = 0$$

$$\therefore am^2 + bm + c = 0$$
 (because $e^{mx} > 0$ for all m)

The 2 values of m required are the roots of the quadratic equation $am^2 + b m + c = 0$. This equation is called the <u>Auxiliary Quadratic Equation</u> and it may have..

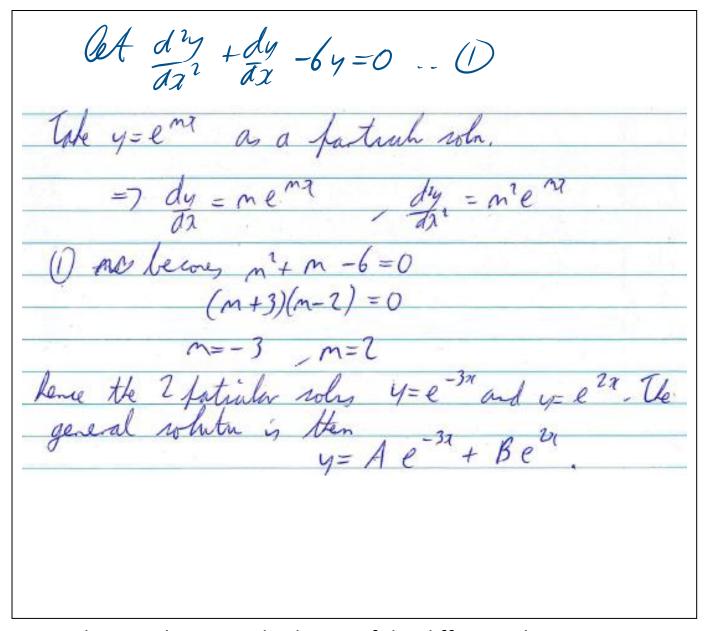
- (i) Real roots (if $b^2 4ac > 0$)
- (ii) Identical roots (if $b^2 4ac = 0$)
- (iii) Complex roots (if $b^2 4ac < 0$)

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Solution



Generalising:- The general solution of the differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, whose auxiliary quadratic equation $am^2+bm+c=0$ has real distinct roots α and β is:-

$$y = Ae^{\alpha x} + Be^{\beta x}$$

(where A and B are constants)

*P3 Book Ex8D

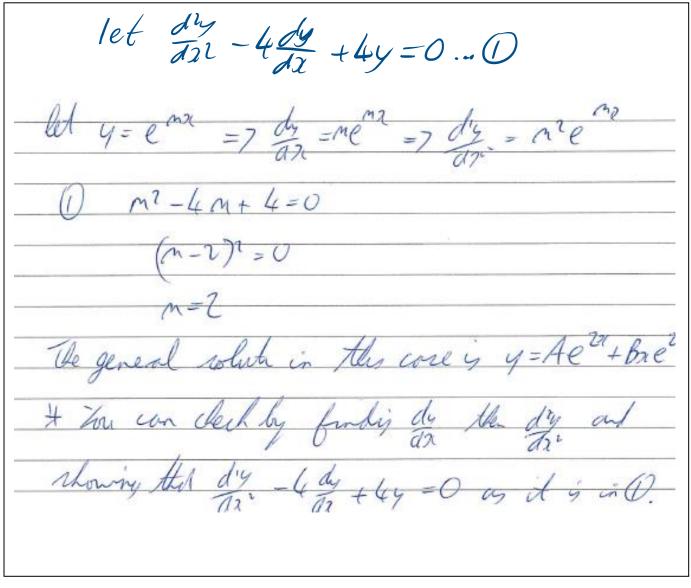
Auxiliary Quadratic Equation With Real Coincident Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Solution



Generalising:- The general solution of the differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, whose auxiliary quadratic equation $am^2+b\ m+c=0$ has equal roots α is:-

$$y = (A + Bx)e^{\alpha x}$$

(where A and B are constants)

Auxiliary Quadratic Equation With Pure Imaginary Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

Solution

$\frac{d^2y}{da^2} + 4y = 0.0$
let 4= en = > dy = ne mot => dry = no emot
(1) $m^2 + 4 = 0$ $m^2 = -4$
$m = \pm 2i$
gen soln is y= Aetia + Be-lix
=7 y = A (cos 20(+ i sin 201) + B (cos 2x - i sin 201)
=7 y= (A+B) co 22 +i(A-B) sin 22
=7 y=Pco 2x + Q Sin 2x when P=A+B Q=i (A-B)

Result:- For the differential equation

$$\frac{d^2y}{dx^2} + n^2y = 0$$

General solution is $y = A \cos nx + B \sin nx$ (where A and B are constants).

Auxiliary Quadratic Equation With Complex Conjugate Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

Solution

diy - 4 dy + 134=0 - (1)	
let y = e ma	
=7 O becomes m2-4m + 13=0	
$=7 M = 4 \pm \sqrt{16 - 4(13)7}$	
$m = \frac{4 \pm 6i}{2}$	
$m = 2 \pm 3i$	
gen soh is y-Pe(2+3i)x +Qe(2-3i)x	
=> y= e2x [Pe3ix + Qe-3ix]	
y = e 2x (A co3x + B sin3x)	
$ \begin{array}{ll} \mathcal{O}_{a} \cdot \mathcal{U} \\ \mathcal{O}_{a} \cdot \mathcal{U} \end{array} $ $ \begin{array}{ll} \mathcal{O}_{a} \cdot \mathcal{U} \\ \mathcal{O}_{a} \cdot \mathcal{U} \end{array} $	

Result:- For the differential equation

 $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where the auxiliary quadratic equation $am^2 + b m + c = 0$ has complex conjugate roots p + iq and p - iq (where p and $q \in R$) the general solution is $y = e^{Px}(A\cos qx + B\sin qx)$ (where A and B are constants)

*P3 book Ex8F

The Second Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

To solve this type of differential equation:-

Method:-

1. Solve the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

The solution is called the *complementary function*.

2. Find a solution of the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

where f(x) could be any one of these forms: –

- (i) A constant k
- (ii) A linear function px + q
- (iii) An exponential function ke^{px}
- (iv) A trig function e.g. $p \sin x$, $q \cos 2x$ or $p \sin 3x + q \cos 3x$

A solution of the differential equation for any of the forms of f(x) given above can be found by inspection.

This solution, when found, is called a *particular integral* of the equation.

3. The general solution of the diifferential equation is then

$$y = C.F. + P.I.$$

Examples on finding the P.I.

Example

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x)$$

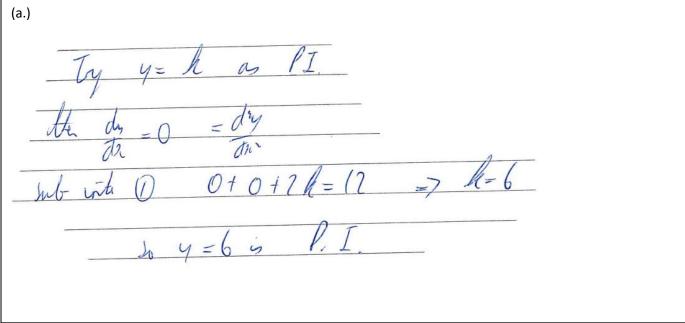
Find P.I. of this differential equation in the cases where $f(x) = \cdots$

(b.)
$$3x + 5$$
 (c.) $3e^{2x}$

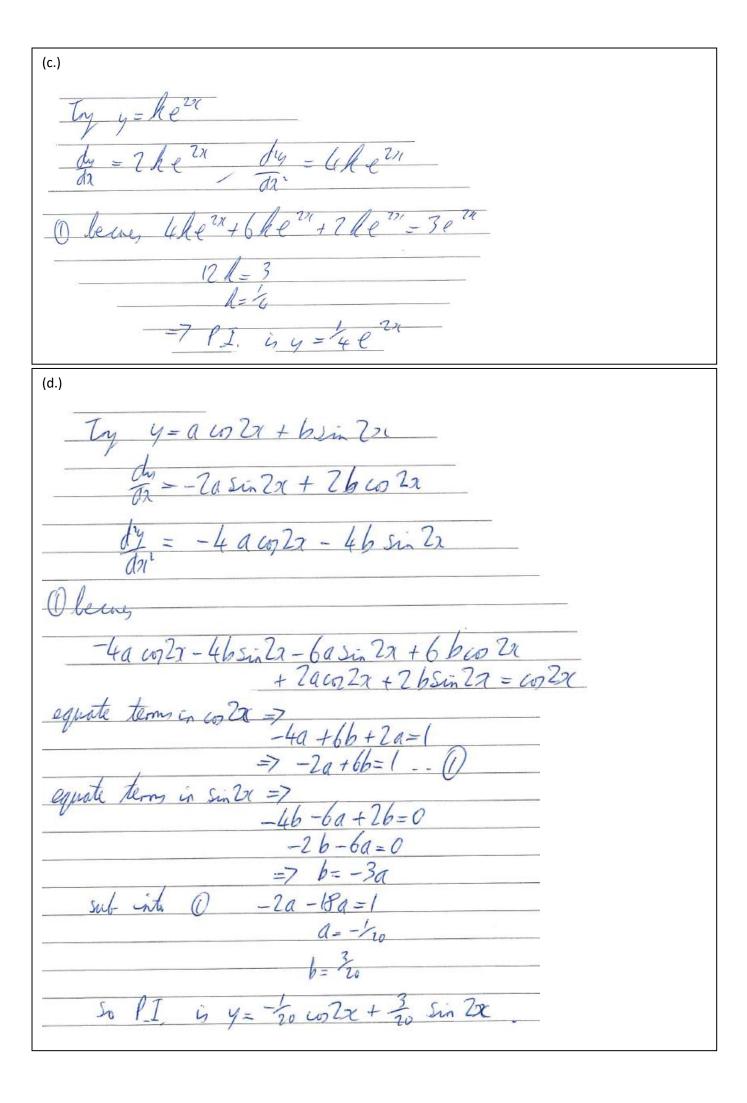
(c.)
$$3e^{2x}$$

(d.)
$$\cos 2x$$

Solution



(b.) becomes 0 + 3a + 3ax + 7b = 32+5 => PI is y=1-5x+0-25



Example

Find y in terms of x for the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cos 2x$$

given that $\frac{dy}{dx} = 0$ at x = 0 and y = 0 at x = 0.

Consider dy + 3 dy + 2y = 0

Auxiliany lips is
$$n^2 + 3m + 2 - 0$$

=7 (m+2)(n+1)=0

m=-2 m=-1

Lence (F is y=Ae 24 + Be 24

So great whis question II is $y=^{-1}v \cos 2x + \frac{3}{20} \sin 2x$
 $x = 0$
 $x =$

^{*}P3 book <u>Ex8G</u> Q1-9odds,17,20,23,26-30