Finding the general equation of a first order differential equation in which the variables are separable.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\therefore \frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\therefore \int \frac{1}{g(y)}dy = \int f(x)dx + c$$

Example

Given that $y = 2$ at $x = 0$ and $\frac{dy}{dx} = y^2 + 4$, find y in terms of x.

Solution

$$\frac{dy}{dx} = y^2 + 4$$

$$\Rightarrow \int \frac{1}{y^2+4}\,dy = \int dx$$

$$\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = x + c$$

when $x=0$, $y=2$ $\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right) = c$

$$\frac{1}{2}\tan^{-1}(1) = c$$

$$\frac{1}{2}\left(\frac{\pi}{4}\right) = c$$

$$c = \cancel{\tfrac{\pi}{8}}\ \frac{1}{2}\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{2}\right) = 2x + \frac{\pi}{4}$$

$$\Rightarrow y = 2\tan\left(2x + \frac{\pi}{4}\right)$$

*P3 book Ex8A Q18-22

First Order Linear Differential Equaations

A 1^{st} order linear differential equation is of the form

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of *x* or constants.

An equation of this form is said to be exact when one side is the exact derivative of a product and the other side can be integrated wrt *x*.

If it is not exact then it can be made exact by multiplying through the equation by a function of *x*. This function is called the integrating factor.

Example

Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

Multiplying through by *x* gives…

$$x\frac{dy}{dx} + y = x^3$$

…making it exact.

$$\therefore \frac{d(xy)}{dx} = x^3$$

$$\therefore xy = \int x^3 dx$$

$$\therefore xy = \frac{x^4}{4} + c$$

In this case the integrating factor is *x*.

Note:- The integrating factor is given by $f(x)$ where $f(x) = e^{\int Pdx}$.

i.e. in the last example $P = \frac{1}{x}$

$$\therefore f(x) = e^{\int \frac{1}{x} dx}$$

$$\therefore f(x) = e^{\ln x}$$

$$\therefore f(x) = x$$

So the linear equation $\frac{dy}{dx} + Py = Q$ can be solved by multiplying by the integgrating factor $e^{\int P dx}$, provided $e^{\int P dx}$ can be found and the function $Qe^{\int P dx}$ can be integrated wrt *x*.

<u>Example</u>

Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = \sin x \cos^3 x$$

<u>Solution</u>

$$\cos x \frac{dy}{dx} + y \sin x = \sin x \cos^3 x$$

first divide through by $\cos x$ to get in $\frac{dy}{dx} + Py = Q$ form

$$\Rightarrow \frac{dy}{dx} + \frac{\sin x}{\cos x} y = \sin x \cos^2 x \quad \text{--} \; ①$$

$$IF = e^{\int \frac{\sin x}{\cos x} dx}$$

$$= e^{-\ln(\cos x)} = e^{\ln(\sec x)} = \sec x$$

mult through by $\sec x$

$$\Rightarrow ① \text{ being } \sec x \frac{dy}{dx} + \tan x \sec x \, y = \sin x \cos x$$

$$\Rightarrow \frac{d(y \sec x)}{dx} = \sin x \cos x$$

$$\Rightarrow \frac{d(y \sec x)}{dx} = \frac{1}{2} \sin 2x$$

$$y \sec x = -\frac{1}{4} \cos 2x + c$$

Example

Find y in terms of x given that

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \ln x \;\; for\ x > 0\ and\ y = 2\ at\ x = 1$$

Solution

$IF = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$

$\Rightarrow \quad x^{-2}\frac{dy}{dx} - 2x^{-3}y = \ln x$

$\Rightarrow \quad \frac{d(x^{-2}y)}{dx} = \ln x$

$\Rightarrow \quad x^{-2}y = \int \ln x \, dx$

let $u = \ln x$ then $\frac{du}{dx} = \frac{1}{x}$ | $\frac{dv}{dx} = 1$, $v = x$

$\Rightarrow \quad x^{-2}y = x\ln x - \int x \cdot \frac{1}{x} dx$

$\frac{y}{x^2} = x\ln x - x + c$

when $x = 1$, $y = 2$ $\Rightarrow 2 = 0 - 1 + c \Rightarrow c = 3$

$\Rightarrow \frac{y}{x^2} = x\ln x - x + 3 \Rightarrow y = x^3\ln x - x^3 + 3x^2$

*P3 book Ex8C Q1-9,13,14,16,17,18

The Second Order Linear Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \; where \; a, b \; and \; c \; are \; constants$$

The equation is called the 2nd order, because its highest derivative of y wrt x is $\frac{d^2y}{dx^2}$.

The equation is called linear because only 1st degree terms in y and its derivatives occur.

Result: The general solution of the 2nd order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is $y = Au + Bv$,

where $y = u \; and \; y = v$ are particular, distinct solutions of the differential equation.

We now need to find the functions $u \; and \; v$ in specific cases.

In the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, try as a solution…

$$y = e^{mx} \text{ where } m \text{ is a constant to be found.}$$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

If $y = e^{mx}$ is a solution of the differential equation then

$$am^2e^{mx} + b\,me^{mx} + ce^{mx} = 0$$

$$\therefore am^2 + b\,m + c = 0 \; (because \; e^{mx} > 0 \; for \; all \; m)$$

The 2 values of m required are the roots of the quadratic equation $am^2 + b\,m + c = 0$. This equation is called the Auxiliary Quadratic Equation and it may have..

(i) Real roots $(if \; b^2 - 4ac > 0)$

(ii) Identical roots $(if \; b^2 - 4ac = 0)$

(iii) Complex roots $(if \; b^2 - 4ac < 0)$

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Solution

Let $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$ .. (1)

Take $y = e^{mx}$ as a particular soln.

$\Rightarrow \frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$

(1) now becomes $m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3$, $m = 2$

Hence the 2 particular solns $y = e^{-3x}$ and $y = e^{2x}$. The general solution is then

$y = Ae^{-3x} + Be^{2x}$.

Generalising:- The general solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, whose auxiliary quadratic equation $am^2 + b\,m + c = 0$ has real distinct roots $\alpha\ and\ \beta$ is:-

$$y = Ae^{\alpha x} + Be^{\beta x}$$

(where A and B are constants)

*P3 Book Ex8D

Auxiliary Quadratic Equation With Real Coincident Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Solution

let $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 \dots (1)$

let $y = e^{mx} \Rightarrow \frac{dy}{dx} = me^{mx} \Rightarrow \frac{d^2y}{dx^2} = m^2e^{mx}$

(1) $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2$

The general solution in this case is $y = Ae^{2x} + Bxe^{2x}$

* You can check by finding $\frac{dy}{dx}$ then $\frac{d^2y}{dx^2}$ and

showing that $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ as it is in (1).

Generalising:- The general solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, whose auxiliary quadratic equation $am^2 + b\,m + c = 0$ has equal roots α is:-

$$y = (A + Bx)e^{\alpha x}$$

(where A and B are constants)

*P3 Book Ex8E

Auxiliary Quadratic Equation With Pure Imaginary Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

Solution

$$\frac{d^2y}{dx^2} + 4y = 0 \quad \ldots (1)$$

let $y = e^{mx} \Rightarrow \frac{dy}{dx} = me^{mx} \Rightarrow \frac{d^2y}{dx^2} = m^2 e^{mx}$

(1) $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

gen soln is $y = Ae^{2ix} + Be^{-2ix}$

$\Rightarrow y = A(\cos 2x + i \sin 2x) + B(\cos 2x - i \sin 2x)$

$\Rightarrow y = (A+B)\cos 2x + i(A-B)\sin 2x$

$\Rightarrow y = P\cos 2x + Q\sin 2x$ where $P = A+B$, $Q = i(A-B)$

Result:- For the differential equation

$$\frac{d^2y}{dx^2} + n^2y = 0$$

General solution is $y = A\cos nx + B\sin nx$ (where A and B are constants).

<u>Auxiliary Quadratic Equation With Complex Conjugate Roots</u>

<u>Example</u>

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

<u>Solution</u>

$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0 \quad \text{..(1)}$

let $y = e^{mx}$

$\Rightarrow$ (1) becomes $m^2 - 4m + 13 = 0$

$\Rightarrow m = \frac{4 \pm \sqrt{16 - 4(13)}}{2}$

$m = \frac{4 \pm 6i}{2}$

$m = 2 \pm 3i$

gen soln is $y = Pe^{(2+3i)x} + Qe^{(2-3i)x}$

$\Rightarrow y = e^{2x}\left[Pe^{3ix} + Qe^{-3ix}\right]$

$y = e^{2x}\left(A\cos 3x + B\sin 3x\right]$

where $A = P + Q$

$B = i(P - Q)$

<u>Result</u>:- For the differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where the auxiliary quadratic equation

$am^2 + b\,m + c = 0$ has complex conjugate roots

$p + iq \text{ and } p - iq \text{ (where } p \text{ and } q \in R)$

the general solution is $y = e^{Px}(A\cos qx + B\sin qx)$

(where A and B are constants)

*P3 book <u>Ex8F</u>

The Second Order Differential Equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

To solve this type of differential equation:-

Method:-

1. Solve the differential equation
$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$
The solution is called the *complementary function*.
2. Find a solution of the equation
$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$
$$where\ f(x) could\ be\ any\ one\ of\ these\ forms:-$$
   (i) A constant k
   (ii) A linear function $px+q$
   (iii) An exponential function ke^{px}
   (iv) A trig function e.g. $p\sin x, q\cos 2x\ or\ psin\ 3x+qcos3x$

A solution of the differential equation for any of the forms of $f(x)$ given above can be found by inspection.

This solution, when found, is called a *particular integral* of the equation.

3. The general solution of the diifferential equation is then
$$y=C.F.+P.I.$$

Examples on finding the P.I.

Example

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x)$$

Find P.I. of this differential equation in the cases where $f(x) = \cdots$

(a.) 12 (b.) $3x + 5$ (c.) $3e^{2x}$ (d.) $\cos 2x$

Solution

(a.)

Try $y = k$ as P.I.

then $\frac{dy}{dx} = 0 = \frac{d^2y}{dx^2}$

sub into ① $0 + 0 + 2k = 12 \Rightarrow k = 6$

so $y = 6$ is P.I.

(b.)

try $y = ax + b$ as P.I.

$\frac{dy}{dx} = a$ $\frac{d^2y}{dx^2} = 0$

① becomes $0 + 3a + 2ax + 2b = 3x + 5$

equate x $\Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$

equate constant $\Rightarrow 3a + 2b = 5$

$4.5 + 2b = 5$

$2b = 0.5$

$b = 0.25$

$\Rightarrow$ PI is $y = 1.5x + 0.25$

(c.)

Try $y = ke^{2x}$

$\frac{dy}{dx} = 2ke^{2x}$, $\frac{d^2y}{dx^2} = 4ke^{2x}$

① becomes $4ke^{2x} + 6ke^{2x} + 2ke^{2x} = 3e^{2x}$

$12k = 3$

$k = \frac{1}{4}$

$\Rightarrow$ P.I. is $y = \frac{1}{4}e^{2x}$

(d.)

Try $y = a\cos 2x + b\sin 2x$

$\frac{dy}{dx} = -2a\sin 2x + 2b\cos 2x$

$\frac{d^2y}{dx^2} = -4a\cos 2x - 4b\sin 2x$

① becomes

$-4a\cos 2x - 4b\sin 2x - 6a\sin 2x + 6b\cos 2x$
$+ 2a\cos 2x + 2b\sin 2x = \cos 2x$

equate terms in $\cos 2x \Rightarrow$
$-4a + 6b + 2a = 1$
$\Rightarrow -2a + 6b = 1$ .. ①

equate terms in $\sin 2x \Rightarrow$
$-4b - 6a + 2b = 0$
$-2b - 6a = 0$
$\Rightarrow b = -3a$

sub into ① $-2a - 18a = 1$
$a = -\frac{1}{20}$
$b = \frac{3}{20}$

So P.I. is $y = -\frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$.

Example

Find y in terms of x for the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cos 2x$$

given that $\frac{dy}{dx} = 0\ at\ x = 0\ and\ y = 0\ at\ x = 0.$

Solution

Consider $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

Auxiliary eqn is $m^2 + 3m + 2 = 0$

$\Rightarrow (m+2)(m+1) = 0$

$m = -2,\ m = -1$

Hence CF is $y = Ae^{-2x} + Be^{-x}$

From (d) of previous question PI is $y = -\frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$

So general soln is $y = Ae^{-2x} + Be^{-x} - \frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$

$\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\} \Rightarrow 0 = A + B - \frac{1}{20}$

$A + B = \frac{1}{20}$ ... I

$\frac{dy}{dx} = -2Ae^{-2x} - Be^{-x} + \frac{1}{10}\sin 2x + \frac{3}{10}\cos 2x$

$x = 0,\ \frac{dy}{dx} = 0 \Rightarrow 0 = -2A - B + \frac{3}{10}$

$2A + B = \frac{3}{10}$ ... II

$\Rightarrow A = \frac{5}{20} = \frac{1}{4}$ ... II − I

$\Rightarrow B = -\frac{1}{5}$

$\Rightarrow$ soln of diff eqn is $y = \frac{1}{4}e^{-2x} - \frac{1}{5}e^{-x} - \frac{1}{20}\cos 2x + \frac{3}{20}\sin 2x$

*P3 book Ex8G Q1-9odds,17,20,23,26-30