Complex Numbers 2

1.1 Exponential form of complex numbers

You can use the modulus–argument form of a complex number to express it in the **exponential form**: $z = re^{i\theta}$.

> **Links** The **modulus–argument** form of a complex number is $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$.
> ← Book 1, Section 2.3

We see Maclaurin later on

You can write $\cos\theta$ and $\sin\theta$ as infinite series of powers of θ:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + \frac{(-1)^r\theta^{2r}}{(2r)!} + \ldots \quad (1)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots + \frac{(-1)^r\theta^{2r+1}}{(2r+1)!} + \ldots \quad (2)$$

You can also write e^x, $x \in \mathbb{R}$, as a series expansion in powers of x.

> **Links** These are the Maclaurin series expansions of $\sin\theta$, $\cos\theta$ and e^x.
> → Chapter 2

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^r}{r!} + \ldots$$

You can use this expansion to define the exponential function for complex powers, by replacing x with a complex number. In particular, if you replace x with the imaginary number $i\theta$, you get

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

OR

let $z = \cos\theta + i\sin\theta$

$\frac{dz}{d\theta} = -\sin\theta + i\cos\theta$

$= i(\cos\theta + i\sin\theta)$

$= iz$

$\Rightarrow \frac{dz}{d\theta} = iz$

$\int \frac{1}{z}\,dz = \int i\,d\theta$

$\ln z = i\theta + c$

when $\theta = 0$ $z = 1$

$\Rightarrow \ln 1 = i(0) + c$

$0 = c$

$\Rightarrow \ln z = i\theta$

$\Rightarrow z = e^{i\theta}$

By comparing this series expansion with (1) and (2), you can write $e^{i\theta}$ as

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula is known as **Euler's relation**. It is important for you to remember this result.

- **You can use Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$, to write a complex number z in exponential form:**

  $$z = re^{i\theta}$$

  where $r = |z|$ and $\theta = \arg z$.

> **Note** Substituting $\theta = \pi$ into Euler's relation yields **Euler's identity**:
>
> $$e^{i\pi} + 1 = 0$$
>
> This equation links the five fundamental constants 0, 1, π, e and i, and is considered an example of mathematical beauty.

Example 1

Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$.

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$ **b** $z = 5\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

(a) $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$

$r = \sqrt{2}$ $\theta = \frac{\pi}{10}$

$\Rightarrow z = \sqrt{2}\, e^{\frac{\pi}{10} i}$

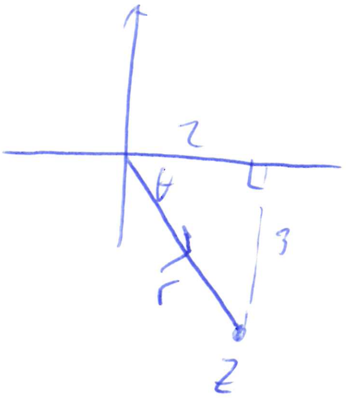
(b) $z = 5\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(\frac{-\pi}{8}\right)\right)$

$\Rightarrow r = 5$ $\theta = \frac{-\pi}{8}$

$\Rightarrow z = 5\, e^{-\frac{\pi}{8} i}$

Example 2

Express $z = 2 - 3i$ in the form $re^{i\theta}$, where $-\pi < \theta \leq \pi$.



$\theta = \tan^{-1}\left(\frac{3}{2}\right)$

$\theta = -0.983\ (3sf)$

$\Rightarrow z = \sqrt{13}\, e^{-0.983 i}$

$r = |z| = \sqrt{2^2 + 3^2}$

$= \sqrt{13}$

Example 3

Express $z = \sqrt{2}e^{\frac{3\pi i}{4}}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$z = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$z = -1 + i$

Example 4

Express $z = 2e^{\frac{23\pi i}{5}}$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

$z = 2\left(\cos\frac{23\pi}{5} + i\sin\frac{23\pi}{5}\right)$ $\quad \frac{23\pi}{5} - 4\pi = \frac{3\pi}{5}$

$z = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

Example 5

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{-i\theta} = \cos\theta - i\sin\theta$ $\quad \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

add $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\Rightarrow \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$

likewise

$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$\Rightarrow i\sin\theta = \frac{1}{2}(e^{i\theta} - e^{-i\theta})$

1.2 Multiplying and dividing complex numbers

You can apply the modulus–argument rules for multiplying and dividing complex numbers to numbers written in exponential form.

Recall that, for any two complex numbers z_1 and z_2,

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Links These results can be proved by considering the numbers z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$ and using the addition formulae for cos and sin. ← **Book 1, Section 2.3**

Applying these results to numbers in exponential form gives the following result:

■ **If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then:**

- $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

Watch out You cannot automatically assume the laws of indices work the same way with complex numbers as with real numbers. This result only shows that they can be applied in these specific cases.

Example 7

Express $\dfrac{2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$ in the form $re^{i\theta}$.

$= \frac{2}{\sqrt{2}}\frac{e^{i\pi/12}}{e^{i5\pi/6}}$

$= \frac{2}{\sqrt{2}}e^{i\left(\pi/12 - 5\pi/6\right)}$

$= \sqrt{2}\, e^{-3\pi/4\, i}$

1.3 De Moivre's theorem

You can use Euler's relation to find powers of complex numbers given in modulus–argument form.

$$\begin{aligned}(r(\cos\theta + i\sin\theta))^2 &= (re^{i\theta})^2 \\ &= re^{i\theta} \times re^{i\theta} \\ &= r^2e^{i2\theta} \\ &= r^2(\cos 2\theta + i\sin 2\theta)\end{aligned}$$

Similarly, $(r(\cos\theta + i\sin\theta))^3 = r^3(\cos 3\theta + i\sin 3\theta)$, and so on.

The generalisation of this result is known as **de Moivre's theorem**:

■ **For any integer n,**

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$$

You can prove de Moivre's theorem quickly using Euler's relation.

$$(r(\cos\theta + i\sin\theta))^n = (re^{i\theta})^n$$
$$= r^n e^{in\theta}$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

This step is valid for any integer exponent n. ← Exercise 1B, Challenge

You can also prove de Moivre's theorem for **positive integer exponents** directly from the modulus–argument form of a complex number using the addition formulae for sin and cos.

Links This proof uses the method of proof by induction. ← Book 1, Chapter 8

1. Basis step

$n = 1$; LHS $= (r(\cos\theta + i\sin\theta))^1 = r(\cos\theta + i\sin\theta)$

RHS $= r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$

As LHS = RHS, de Moivre's theorem is true for $n = 1$.

2. Assumption step

Assume that de Moivre's theorem is true for $n = k$, $k \in \mathbb{Z}^+$:

$(r(\cos\theta + i\sin\theta))^k = r^k(\cos k\theta + i\sin k\theta)$

3. Inductive step

When $n = k + 1$,

$$(r(\cos\theta + i\sin\theta))^{k+1} = (r(\cos\theta + i\sin\theta))^k \times r(\cos\theta + i\sin\theta)$$
$$= r^k(\cos k\theta + i\sin k\theta) \times r(\cos\theta + i\sin\theta) \quad \text{By assumption step}$$
$$= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$
$$= r^{k+1}((\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta))$$
$$= r^{k+1}(\cos(k\theta + \theta) + i\sin(k\theta + \theta)) \quad \text{By addition formulae}$$
$$= r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta))$$

Therefore, de Moivre's theorem is true when $n = k + 1$.

4. Conclusion step

If de Moivre's theorem is true for $n = k$, then it has been shown to be true for $n = k + 1$.

As de Moivre's theorem is true for $n = 1$, it is now proven to be true for all $n \in \mathbb{Z}^+$ by mathematical induction.

Links The corresponding proof for negative integer exponents is left as an exercise. → Exercise 1C, Challenge

Example 8

Simplify $\dfrac{\left(\cos\frac{9\pi}{17} + i\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17} - i\sin\frac{2\pi}{17}\right)^3}$

$$= \frac{\cos 5\left(\frac{9\pi}{17}\right) + i\sin 5\left(\frac{9\pi}{17}\right)}{\cos\left(3\left(\frac{-2\pi}{17}\right)\right) + i\sin 3\left(\frac{-2\pi}{17}\right)}$$

$$= \frac{\cos\frac{45\pi}{17} + i\sin\frac{45\pi}{17}}{\cos\left(\frac{-6\pi}{17}\right) + i\sin\left(\frac{-6\pi}{17}\right)}$$

$$= \cos\left(\frac{45\pi}{17} - \left(\frac{-6\pi}{17}\right)\right) + i\sin\left(\frac{45\pi}{17} - \left(\frac{-6\pi}{17}\right)\right)$$

$$= \cos\frac{51\pi}{17} + i\sin\frac{51\pi}{17}$$

$$= \cos 3\pi + i\sin 3\pi$$

$$= -1 + i(0)$$

$$= -1$$

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Example 10

Use de Moivre's theorem to show that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

$$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta$$

$$\Rightarrow \cos 6\theta + i\sin 6\theta = \cos^6\theta + 6\cos^5\theta(i\sin\theta) + 15\cos^4\theta(i^2\sin^2\theta) + 20\cos^3\theta(i^3\sin^3\theta)$$
$$+ 15\cos^2\theta(i^4\sin^4\theta) + 6\cos\theta(i^5\sin^5\theta) + i^6\sin^6\theta$$
$$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta$$
$$+ 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$$

equate reals

$$\Rightarrow \cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$$
$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$
$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta) - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$
$$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta - 30\cos^4\theta + 15\cos^6\theta - 1 + 3\cos^2\theta$$
$$- 3\cos^4\theta + \cos^6\theta$$

$$\Rightarrow \boxed{\cos 6\theta} = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1 \quad \checkmark$$

You can also find trigonometric identities for $\sin^n\theta$ and $\cos^n\theta$ where n is a positive integer.

If $z = \cos\theta + i\sin\theta$, then

$$\frac{1}{z} = z^{-1} = (\cos\theta + i\sin\theta)^{-1}$$
$$= (\cos(-\theta) + i\sin(-\theta))$$ — Apply de Moivre's theorem.
$$= \cos\theta - i\sin\theta$$ — Use $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$.

It follows that

$$z + \frac{1}{z} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$$

$$z - \frac{1}{z} = \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta) = 2i\sin\theta$$

Also,

$$z^n = (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$ By de Moivre's theorem.

$$\frac{1}{z^n} = z^{-n} = (\cos\theta + \mathrm{i}\sin\theta)^{-n}$$

$$= (\cos(-n\theta) + \mathrm{i}\sin(-n\theta))$$ Apply de Moivre's theorem.

$$= \cos n\theta - \mathrm{i}\sin n\theta$$ Use $\cos\theta = \cos(-\theta)$ and $\sin(-\theta) = -\sin\theta$.

It follows that

$$z^n + \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta + \cos n\theta - \mathrm{i}\sin n\theta = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = \cos n\theta + \mathrm{i}\sin n\theta - (\cos n\theta - \mathrm{i}\sin n\theta) = 2\mathrm{i}\sin n\theta$$

It is important that you remember and are able to apply these results:

- $z + \frac{1}{z} = 2\cos\theta$
- $z^n + \frac{1}{z^n} = 2\cos n\theta$
- $z - \frac{1}{z} = 2\mathrm{i}\sin\theta$
- $z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$

Notation In exponential form, these results are equivalent to:

$\cos n\theta = \frac{1}{2}(e^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta})$ $\sin n\theta = \frac{1}{2\mathrm{i}}(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta})$.

Example 11

Express $\cos^5\theta$ in the form $a\cos 5\theta + b\cos 3\theta + c\cos\theta$, where a, b and c are constants.

let $z = \cos\theta + i\sin\theta \Rightarrow z^{-1} = \cos\theta - i\sin\theta$ and $z^n = \cos n\theta + i\sin n\theta$

$(z + z^{-1}) = 2\cos\theta$ $z^{-n} = \cos n\theta - in\theta$

$\Rightarrow z^n + z^{-n} = 2\cos n\theta$

$(2\cos\theta)^5 = (z + z^{-1})^5$

$= z^5 + 5z^4z^{-1} + 10z^3z^{-2} + 10z^2z^{-3} + 5zz^{-4} + z^{-5}$

$32\cos^5\theta = z^5 + z^{-5} + 5(z^3 + z^{-3}) + 10(z + z^{-1})$

$32\cos^5\theta = 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$

$32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\Rightarrow \cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{5}{8}\cos\theta$

* P4 book Ex3B Q1-6, ~~7~~, 9-12, 23, 24

Example 12

a Express $\sin^4\theta$ in the form $d\cos 4\theta + e\cos 2\theta + f$, where d, e and f are constants.

b Hence find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$.

(a) Let $z = \cos\theta + i\sin\theta \Rightarrow z^{-1} = \cos\theta - i\sin\theta$

$\Rightarrow (z - z^{-1}) = 2i\sin\theta$ and $z + z^{-1} = 2\cos\theta$ & $z^n + z^{-n} = 2\cos n\theta$

$$\begin{array}{c} 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \end{array}$$

$(z - z^{-1})^4 = z^4 + 4z^3(-z^{-1}) + 6z^2(-z^{-1})^2 + 4z(-z^{-1})^3 + (z^{-1})^4$

$= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$

$= z^4 + z^{-4} - 4(z^2 + z^{-2}) + 6$

$\Rightarrow (2i\sin\theta)^4 = 2\cos 4\theta - 4(2\cos 2\theta) + 6$

$\Rightarrow 16\sin^4\theta = 2\cos 4\theta - 8\cos 2\theta + 6$

$\Rightarrow \sin^4\theta = \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}$

(b) $\int_0^{\pi/2} \sin^4\theta \, d\theta = \int_0^{\pi/2} \frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8} \, d\theta$

$= \left[\frac{1}{32}\sin 4\theta - \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta\right]_0^{\pi/2}$

$= \left[\frac{1}{32}\sin 2\pi - \frac{1}{4}\sin\pi + \frac{3\pi}{16}\right] - [0]$

$= 0 - 0 + 3\pi/16$

$= 3\pi/16$

*P4 book Ex 3B Q13 b.

1.6 nth roots of a complex number

You can use de Moivre's theorem to solve an equation of the form $z^n = w$, where $z, w \in \mathbb{C}$. This is equivalent to finding the nth roots of w.

Just as a real number, x, has two square roots, $\sqrt{x}$ and $-\sqrt{x}$, any complex number has n distinct nth roots.

- **If z and w are non-zero complex numbers and n is a positive integer, then the equation $z^n = w$ has n distinct solutions.**

You can find the solutions to $z^n = w$ using de Moivre's theorem, and by considering the fact that the argument of a complex number is not unique.

Note $\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$ for integer values of k.

- **For any complex number $z = r(\cos\theta + i\sin\theta)$, you can write $z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$, where k is any integer.**

Example 15

a Solve the equation $z^3 = 1$.

b Represent your solutions to part **a** on an Argand diagram.

c Show that the three cube roots of 1 can be written as 1, ω and ω^2 where $1 + \omega + \omega^2 = 0$.

(a) $z^3 = 1$

$z^3 = \cos 0 + i\sin 0$ ← already in modulus, argument form.

which you can write as

$z^3 = \cos(0 + 2k\pi) + i\sin(0 + 2k\pi)$

where $k \in \mathbb{Z}$.

$z = \left(\cos(0 + 2k\pi) + i\sin(0 + 2k\pi)\right)^{\frac{1}{3}}$

$z = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3}$

Let $k=0 \Rightarrow z_1 = \cos 0 + i\sin 0 = 1$

$k=1 \Rightarrow z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

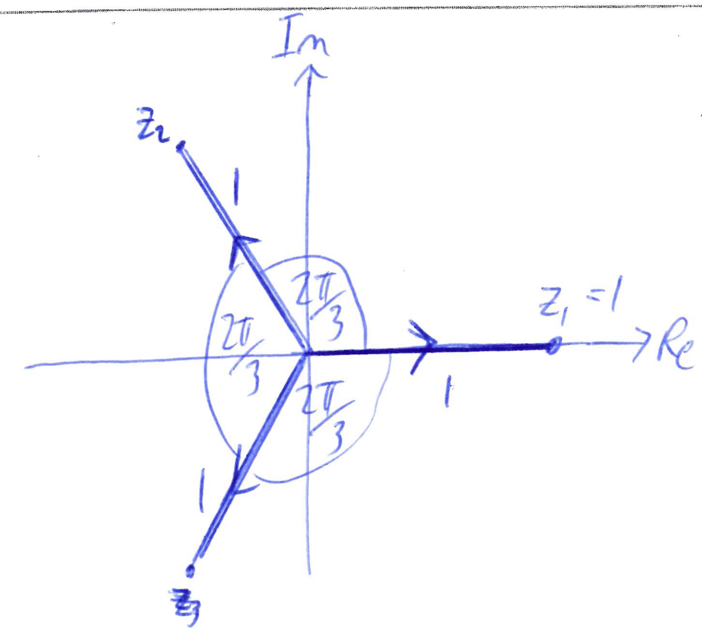
$k=2 \Rightarrow z_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$= \cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3}) = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

these are the cube roots of unity.

Note if you go beyond $k=2$ you are just repeating the same roots.

(b)



the pts. z_1, z_2 and z_3 lie on a circle of a radius 1 unit. The angle between each is $2\pi/3$ c.

(c) let $w = z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{2\pi}{3}i}$

$w^2 = \left(e^{\frac{2\pi}{3}i}\right)^2 = e^{\frac{4\pi}{3}i} = e^{-\frac{2\pi}{3}i}$

$= -\frac{1}{2} - i\frac{\sqrt{3}}{2} = z_3$

$1 + w + w^2$

$= 1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 0$ ✓

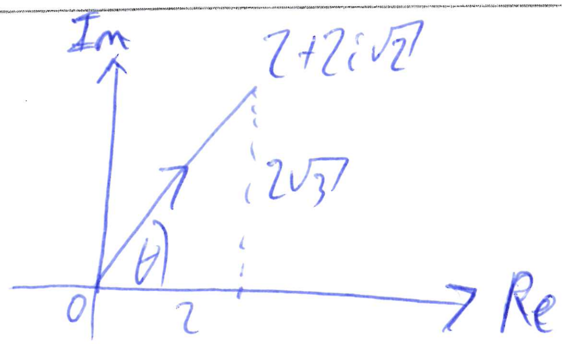
- **In general, the solutions to $z^n = 1$ are $z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) = e^{\frac{2\pi i k}{n}}$ for $k = 1, 2, \ldots, n$ and are known as the nth roots of unity.**

If n is a positive integer, then there is an nth root of unity $\omega = e^{\frac{2\pi i}{n}}$ such that:

- **the nth roots of unity are $1, \omega, \omega^2, \cdots, \omega^{n-1}$**
- **$1, \omega, \omega^2, \cdots, \omega^{n-1}$ form the vertices of a regular n-gon**
- **$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$**

Example 16

Solve the equation $z^4 = 2 + 2i\sqrt{3}$.



$z^4 = 2 + 2i\sqrt{3}$

$$\text{modulus} = \sqrt{(2)^2 + (2\sqrt{3})^2}$$
$$= 4$$

$$\text{argument} = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

$$\Rightarrow z^4 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\text{or} \quad z^4 = 4\left[\cos\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\left(\frac{\pi}{3} + 2k\pi\right)\right]$$

$$\Rightarrow z = 4^{\frac{1}{4}}\left[\cos\left(\frac{\frac{\pi}{3} + 2k\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{3} + 2k\pi}{4}\right)\right]$$

$$z = \sqrt{2}\left(\cos\left(\frac{\frac{\pi}{3} + 2k\pi}{4}\right) + i\sin\left(\frac{\frac{\pi}{3} + 2k\pi}{4}\right)\right)$$

let $k = 0, 1, 2, 3$

$k=0 \quad z_1 = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$k=1 \quad z_2 = \sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

$k=2 \quad z_3 = \sqrt{2}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$

$\qquad = \sqrt{2}\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$

$k=3 \quad z_4 = \sqrt{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$

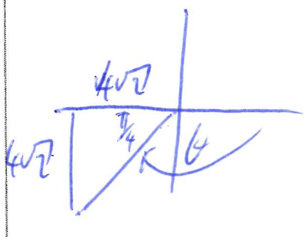
$\qquad = \sqrt{2}\left(\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right)$

<u>OR</u> $z = \sqrt{2}e^{\frac{\pi i}{12}}, \sqrt{2}e^{\frac{7\pi i}{12}}, \sqrt{2}e^{-\frac{11\pi}{12}i}, \sqrt{2}e^{-\frac{5\pi}{12}i}$

Example 17

Solve the equation $z^3 + 4\sqrt{2} + 4i\sqrt{2} = 0$.

rearrange $z^3 = -4\sqrt{2} - 4i\sqrt{2}$



$$\text{modulus} = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2}$$
$$= \sqrt{64}$$
$$= 8$$

$$\arg = -(\pi - \tfrac{\pi}{4})$$
$$\arg = -\tfrac{3\pi}{4}$$

$$\Rightarrow z^3 = 8\left(\cos\left(-\tfrac{3\pi}{4}\right) + i\sin\left(-\tfrac{3\pi}{4}\right)\right) = 8\left(\cos\left(\tfrac{-3\pi}{4} + 2k\pi\right) + i\sin\left(\tfrac{-3\pi}{4} + 2k\pi\right)\right)$$

$$z = 2\left(\cos\left(\frac{-\frac{3\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{-\frac{3\pi}{4} + 2k\pi}{3}\right)\right) = 2e^{i\left(\frac{-\frac{3\pi}{4} + 2k\pi}{3}\right)}$$

let $k = 0, 1, 2$

$k = 0 \quad z = 2e^{-\frac{\pi}{4}i}$

$k = 1 \quad z = 2e^{\frac{5\pi}{12}i}$

$k = 2 \quad z = 2e^{\frac{13\pi}{12}i} = 2e^{-\frac{11\pi}{12}i}$

OR could be written

$$z_1 = 2\left(\cos\left(-\tfrac{\pi}{4}\right) + i\sin\left(-\tfrac{\pi}{4}\right)\right)$$
$$z_2 = 2\left(\cos\tfrac{5\pi}{12} + i\sin\tfrac{5\pi}{12}\right)$$
$$z_3 = 2\left(\cos\left(-\tfrac{11\pi}{12}\right) + i\sin\left(-\tfrac{11\pi}{12}\right)\right)$$

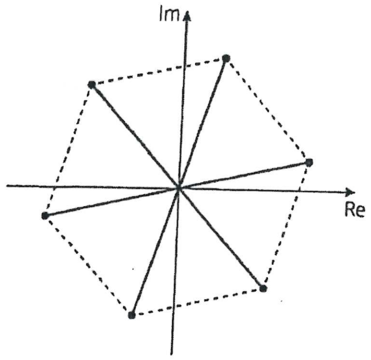
1.7 Solving geometric problems

You can use properties of complex nth roots to solve geometric problems.

- **The nth roots of any complex number a lie at the vertices of a regular n-gon with its centre at the origin.**

The orientation and size of the regular polygon will depend on a.

Notation The centre of a regular polygon is considered to be the centre of the circle that passes through all of its vertices.



For example, the sixth roots of 7 + 24i form this regular hexagon. Each vertex of the hexagon is equidistant from the origin, which lies at the centre of the circle passing through all six vertices.

Online Explore nth roots of complex numbers in an Argand diagram using GeoGebra.

You can find the vertices of this regular polygon by finding a single vertex, and rotating that point around the origin. This is equivalent to multiplying by the nth roots of unity.

- **If z_1 is one root of the equation $z^n = s$, and 1, ω, ω^2, …, ω^{n-1} are the nth roots of unity, then the roots of $z^n = s$ are given by z_1, $z_1\omega$, $z_1\omega^2$, …, $z_1\omega^{n-1}$.**

Example 18

The point $P(\sqrt{3}, 1)$ lies at one vertex of an equilateral triangle. The centre of the triangle is at the origin.

a Find the coordinates of the other vertices of the triangle.

b Find the area of the triangle.

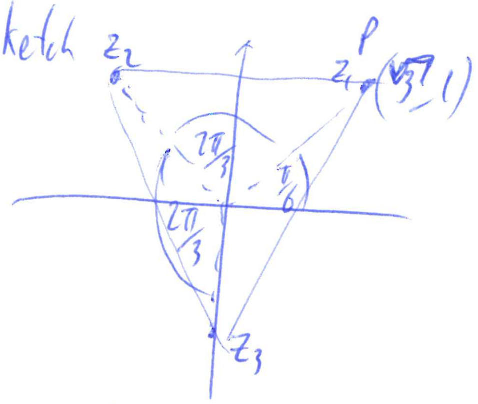
(a) pt P $z = \sqrt{3} + i$

mod $= \sqrt{3+1} = 2$

arg $z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

at P $z = 2e^{i\frac{\pi}{6}}$

sketch



as centre of triangle is origin then you rotate through $\frac{2\pi}{3}^c$ to get the next roots.

so $z_2 = 2e^{i\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)}$

$z_2 = 2e^{i\frac{5\pi}{6}}$

and $z_3 = 2e^{i\left(\frac{5\pi}{6} + \frac{2\pi}{3}\right)}$

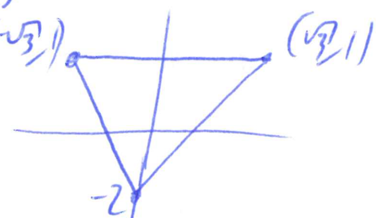
$= 2e^{i\frac{3\pi}{2}} = 2e^{-i\frac{\pi}{2}}$

so $z_1 = \sqrt{3} + i$

$z_2 = -\sqrt{3} + i$

$z_3 = -2$

(b)



Area $= \frac{1}{2} \times$ base $\times$ height

$= \frac{1}{2} \times 2\sqrt{3} \times 3$

$= 3\sqrt{3}$ units^2

* Ex 3B Q ~~13, 15~~ 14-17, ~~23, 24~~ & 18-24

Summary of key points

1 You can use **Euler's relation**, $e^{i\theta} = \cos\theta + i\sin\theta$, to write a complex number z in exponential form:

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg z$.

2 For any two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$,

- $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

3 **De Moivre's theorem:**
For any integer n, $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

4
- $z + \dfrac{1}{z} = 2\cos\theta$
- $z^n + \dfrac{1}{z^n} = 2\cos n\theta$
- $z - \dfrac{1}{z} = 2i\sin\theta$
- $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

5 For $w, z \in \mathbb{C}$,

- $\displaystyle\sum_{r=0}^{n-1} wz^r = w + wz + wz^2 + \ldots + wz^{n-1} = \frac{w(z^n - 1)}{z - 1}$
- $\displaystyle\sum_{r=0}^{\infty} wz^r = w + wz + wz^2 + \ldots = \frac{w}{1 - z}, \ |z| < 1$

6 If z and w are non-zero complex numbers and n is a positive integer, then the equation $z^n = w$ has n distinct solutions.

7 For any complex number $z = r(\cos\theta + i\sin\theta)$, you can write

$$z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$$

where k is any integer.

8 In general, the solutions to $z^n = 1$ are $z = \cos\left(\dfrac{2\pi k}{n}\right) + i\sin\left(\dfrac{2\pi k}{n}\right) = e^{\frac{2\pi ik}{n}}$ for $k = 1, 2, \ldots, n$ and are known as the nth roots of unity.
If n is a positive integer, then there is an nth root of unity $\omega = e^{\frac{2\pi i}{n}}$ such that:

- The nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$
- $1, \omega, \omega^2, \ldots, \omega^{n-1}$ form the vertices of a regular n-gon
- $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$

9 The nth roots of any complex number s lie on the vertices of a regular n-gon with its centre at the origin.

10 If z_1 is one root of the equation $z^n = s$, and $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the nth roots of unity, then the roots of $z^n = s$ are given by $z_1, z_1\omega, z_1\omega^2, \ldots, z_1\omega^{n-1}$.