Further Maths A21 Question Booklet

Complex Numbers Question Booklet A21

-P4 book P92 Ex3A Q1abcegjk,2aegi,3,7 (*A2)

Exercise 3A

1 Write the following in the form (i) $r(\cos\theta + \text{i}\sin\theta)$ (ii) $r\text{e}^{\text{i}\theta}$, $-\pi < \theta \leqslant \pi$, giving θ either as a multiple of π or in radians to 3 significant figures.

(a) 5i (b) 7 (c) -3i (d) -6

(e) $1 + \text{i}\sqrt{3}$ (f) $3\sqrt{3} - 3\text{i}$ (g) $-3 + 4\text{i}$ (h) $1 - \text{i}$

(i) $6 - 8\text{i}$ (j) $\dfrac{2}{1 - \text{i}\sqrt{3}}$ (k) $\dfrac{8}{\sqrt{3} - \text{i}}$ (l) $\dfrac{3 - 2\text{i}}{1 + 4\text{i}}$

2 Write the following in the form $a + \text{i}b$, $a, b \in \mathbb{R}$:

(a) $3\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)$ (b) $-5\left(\cos\frac{\pi}{4} - \text{i}\sin\frac{\pi}{4}\right)$

(c) $6\left[\cos\left(-\frac{\pi}{6}\right) + \text{i}\sin\left(-\frac{\pi}{6}\right)\right]$ (d) $-4\left(\cos\frac{3\pi}{2} + \text{i}\sin\frac{3\pi}{2}\right)$

(e) $2\left(\cos\frac{2\pi}{7} + \text{i}\sin\frac{2\pi}{7}\right) \times 5\left(\cos\frac{5\pi}{7} + \text{i}\sin\frac{5\pi}{7}\right)$

(f) $\left[3\left(\cos\frac{7\pi}{12} + \text{i}\sin\frac{7\pi}{12}\right)\right]^2$ (g) $\dfrac{7\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)}{3\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)}$

(h) $\dfrac{6\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)}{2\left(\cos\frac{\pi}{2} - \text{i}\sin\frac{\pi}{2}\right)}$ (i) $\left[2\left(\cos\frac{5\pi}{18} + \text{i}\sin\frac{5\pi}{18}\right)\right]^3$

(j) $\dfrac{\left[2\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)\right]^2}{3\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)}$

3 Simplify, without the use of a calculator

$$\frac{\left(\cos\frac{\pi}{7} - \text{i}\sin\frac{\pi}{7}\right)^3}{\left(\cos\frac{\pi}{7} + \text{i}\sin\frac{\pi}{7}\right)^4}$$

[L]

7 Use the fact that $\text{e}^x\sin 3x = \text{Im}\left(\text{e}^x\text{e}^{\text{i}3x}\right)$ to find

$$\int \text{e}^x \sin 3x \, \text{d}x$$

Exercise 3A

1 (a) (i) $5\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$
(ii) $5e^{\frac{i\pi}{2}}$

(b) (i) $7(\cos 0 + \mathrm{i}\sin 0)$
(ii) $7e^{0}$

(c) (i) $3\left[\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right]$
(ii) $3e^{-\frac{i\pi}{2}}$

(d) (i) $6(\cos\pi + \mathrm{i}\sin\pi)$
(ii) $6e^{i\pi}$

(e) (i) $2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$
(ii) $2e^{\frac{i\pi}{3}}$

(f) (i) $6\left[\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]$
(ii) $6e^{-\frac{i\pi}{6}}$

(g) (i) $5(\cos 2.21 + \mathrm{i}\sin 2.21)$
(ii) $5e^{2.21i}$

(h) (i) $\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$
(ii) $\sqrt{2}e^{-\frac{i\pi}{4}}$

(i) (i) $10[\cos(-0.927) + \mathrm{i}\sin(-0.927)]$
(ii) $10e^{-0.927i}$

(j) (i) $\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$
(ii) $e^{\frac{i\pi}{3}}$

(k) (i) $2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)$
(ii) $2e^{\frac{i\pi}{6}}$

(l) (i) $\frac{\sqrt{221}}{17}[\cos(-1.91) + \mathrm{i}\sin(-1.91)]$
(ii) $\frac{\sqrt{221}}{17}e^{-1.91i}$

2 (a) $\frac{3}{2}(1 + \mathrm{i}\sqrt{3})$ (b) $-\frac{5}{\sqrt{2}} + \frac{5i}{\sqrt{2}}$
(c) $3\sqrt{3} - 3i$ (d) $4i$
(e) -10 (f) $-\frac{9}{2}(\sqrt{3} + i)$
(g) $\frac{7}{3\sqrt{2}}(1 + i)$ (h) $\frac{3}{\sqrt{2}}(-1 + i)$
(i) $4(-\sqrt{3} + i)$ (j) $\frac{2}{3}(\sqrt{3} + i)$

3 $e^{-\pi i} = -1$

4 $\sinh z \cosh w + \cosh z \sinh w$

5 $\dfrac{\tanh z + \tanh w}{1 + \tanh z \tanh w}$

6 $1 - \tanh^2 z$

7 $\frac{1}{10}e^{x}(\sin 3x - 3\cos 3x) + C$

-P4 book P103 Ex3B Q1-6, 9-12,13b, 14-17, 23, 24 *also Q 18-22

Exercise 3B

1 Use de Moivre's theorem to simplify:

(a) $\left[\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right]^{10}$ (b) $\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]^{8}$

(c) $\left[\cos\left(-\frac{\pi}{27}\right) + i\sin\left(-\frac{\pi}{27}\right)\right]^{9}$ (d) $\left(\cos\frac{\pi}{18} - i\sin\frac{\pi}{18}\right)^{3}$

2 Express $z = 2(1 - i\sqrt{3})$ in the form $r(\cos\theta + i\sin\theta)$.
Hence find z^8 and $\frac{1}{z^5}$ in the form $a + ib$.

3 Express $z = (1 - i)$ in the form $r(\cos\theta + i\sin\theta)$. Hence find z^4 and $\frac{1}{z^7}$ in the form $a + ib$.

4 Simplify $\dfrac{\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^4}{\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)^5}$ [L]

5 Find $\sin 5\theta$ in terms of $\sin\theta$.

6 Find $\sin 3\theta$ in terms of $\sin\theta$.

7 Find $\cos 7\theta$ in terms of $\cos\theta$.

8 Find $\sin 7\theta$ in terms of $\sin\theta$.

9 Find $\tan 3\theta$ in terms of $\tan\theta$.

10 Find $\tan 5\theta$ in terms of $\tan\theta$.

11 Express (a) $\cos^5\theta$ (b) $\cos^6\theta$ (c) $\cos^7\theta$ in terms of cosines of multiples of θ.

12 Express (a) $\sin^4\theta$ in terms of cosines of multiples of θ
(b) (i) $\sin^5\theta$ (ii) $\sin^7\theta$ in terms of sines of multiples of θ.

13 Find (a) $\int \sin^4\theta \, d\theta$ (b) $\int \cos^6\theta \, d\theta$ (see 11b) (c) $\int \sin^4\theta\cos^2\theta \, d\theta$.

14 Find, in the form $re^{i\theta}$, the cube roots of:

(a) i (b) -1 (c) $-5+12i$ (d) $\frac{1+i}{1-i}$

15 Solve the equation

$$z^5+1=0$$

16 Find the cube roots of $21+72i$.

17 Find the fourth roots of unity.

18 One root of the equation $2z^3-9z^2+30z-13=0$ is $2+3i$. Find the other two roots.

19 One root of the equation $z^3-10z^2+33z-34=0$ is $4+i$. Find the other two roots.

20 One root of the equation $2z^3-5z^2+12z-5=0$ is $1-2i$. Find the other two roots.

21 One root of the equation $z^4+3z^3+12z-16=0$ is $2i$. Find the other three roots.

22 One root of the equation $2z^4-11z^3+27z^2-25z+7=0$ is $2-i\sqrt{3}$. Find the other three roots.

23 If $2\cos\theta = z+z^{-1}$, prove that, if n is a positive integer,

$$2\cos n\theta = z^n + z^{-n}$$

Hence, or otherwise, solve the equation

$$3z^4-z^3+2z^2-z+3=0$$

given that no root is real. [L]

24 If $z=\cos\theta + i\sin\theta$, show that

$$z^n+z^{-n}=2\cos n\theta \quad \text{and} \quad z^n-z^{-n}=2i\sin n\theta$$

Hence deduce that

$$\cos^6\theta+\sin^6\theta=\tfrac{1}{8}(3\cos 4\theta+5)$$ [L]

ANSWERS

Exercise 3B

1 (a) $\cos 4\pi + i\sin 4\pi = 1$
(b) $\cos\frac{8\pi}{12} + i\sin\frac{8\pi}{12} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$
(c) $\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$
(d) $\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

2 $4\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$; $4^8\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$; $\frac{1}{4^5}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

3 $\sqrt{2}[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$; -4; $\frac{1}{16}(1-i)$

4 $\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^9 = \cos\pi + i\sin\pi = -1$

5 $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

6 $3\sin\theta - 4\sin^3\theta$

7 $-7\cos\theta + 56\cos^3\theta - 112\cos^5\theta + 64\cos^7\theta$

8 $7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

9 $\dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

10 $\dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

11 (a) $\frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$
(b) $\frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$
(c) $\frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$

12 (a) $\frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$
(b) $\frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$
(c) $-\frac{1}{64}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta \mp 35\sin\theta)$

13 (a) $\frac{1}{8}\left(\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta\right) + C$
(b) $\frac{1}{32}\left(\frac{1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right) + C$
(c) $\frac{1}{192}\sin 6\theta + \frac{5}{64}\sin 4\theta - \frac{1}{64}\sin 2\theta + \frac{\theta}{16} + C$

14 (a) $e^{i\left(\frac{2k\pi}{3} + \frac{\pi}{6}\right)}$, $k = 0, 1, 2$
(b) $e^{i\left(\frac{2k\pi}{3} + \frac{\pi}{3}\right)}$, $k = 0, 1, 2$
(c) $\sqrt[3]{(13)}e^{i\left(\frac{2k\pi}{3} + \frac{\alpha}{3}\right)}$ where $\alpha = 1.965$
(d) $e^{i\left(\frac{2k\pi}{3} + \frac{\pi}{6}\right)}$, $k = 0, 1, 2$

15 $z = e^{\left(\frac{2k\pi}{5} + \frac{\pi}{5}\right)i}$, $k = 0, 1, 2, 3, 4$

16 $\sqrt[3]{75}\,e^{i\left(\frac{1.287 + 2k\pi}{3}\right)}$, $k = 0, 1, 2$

17 $1, -1, i, -i$

18 $2 - 3i$, $\frac{1}{2}$

19 $4 - i, 2$

20 $1 + 2i$, $\frac{1}{2}$

21 $-2i$, -4, 1

22 $2 + i\sqrt{3}$, $\frac{1}{2}$, 1

23 $\dfrac{2 \pm i\sqrt{5}}{3}$, $-\dfrac{1 \pm i\sqrt{3}}{2}$

Partial Fractions

-P2 book P32 Ex1D Q6-10, 12, 23,24, 28,29,31,33,35

Exercise 1D

Express as partial fractions:

1 $\frac{2x+5}{(x+2)(x+3)}$

2 $\frac{2x+2}{(x-1)(x+3)}$

3 $\frac{x+1}{(x+3)(x+4)}$

4 $\frac{x+7}{x^2+5x+6}$

5 $\frac{2x^2+12x-10}{(x-1)(2x-1)(x+3)}$

6 $\frac{3x^2-x+6}{(x^2+4)(x-2)}$

7 $\frac{x^2-2x+9}{(x^2+3)(x-3)}$

8 $\frac{-2x^2+4x-4}{(x^2+5)(2x+3)}$

9 $\frac{-6x^2+x-12}{(5+2x^2)(x+3)}$

10 $\frac{-2x^2+13}{(2x+1)(x^2+2x+7)}$

11 $\frac{2x-7}{(x-5)^2}$

12 $\frac{x^2+4x+7}{(x+3)^3}$

13 $\frac{-3x^2+10x+5}{(x+2)(x-1)^2}$

14 $\frac{-5x^2+8x+9}{(x+2)(x-1)^2}$

15 $\frac{10x+9}{(2x+1)(2x+3)^2}$

16 $\frac{x}{x-1}$

17 $\frac{x^2}{x-1}$

18 $\frac{x^2+1}{x^2-1}$

19 $\frac{x^2+2}{x(x-1)}$

20 $\frac{x^3}{x^2-1}$

21 $\dfrac{9-2x-2x^2}{(1+x)(2-x)}$

22 $\dfrac{4x^2-3x+2}{2x^2-x-1}$

23 $\dfrac{2x^3+10x^2+12x+1}{(x+2)(x+3)}$

24 $\dfrac{x^3+x^2-2x+4}{x^2-4}$

25 $\dfrac{-x^4-x^3+2x^2-x-2}{x^2(x+1)}$

26 $\dfrac{13}{(2x-3)(3x+2)}$

27 $\dfrac{4x^2+5x+9}{(2x-1)(x+2)^2}$

28 $\dfrac{x^3+4x^2+3x+4}{(x^2+1)(x+1)^2}$

29 $\dfrac{4x^4+6x^3+4x^2+x-3}{x^2(2x+3)}$

30 $\dfrac{4x+3}{(2x-1)(3x+1)}$

31 $\dfrac{x^3+3x^2-2x-5}{(x-1)^2(x^2+2)}$

32 $\dfrac{3x^2+12x+8}{(2x+3)(x^2-4)}$

33 $\dfrac{x^3-x^2-1}{x(x^2+x+1)}$

34 $\dfrac{x^2+2x+3}{x^2(x+1)}$

35 $\dfrac{3}{x^3+1}$

Answers

xercise 1D

1 $\frac{1}{x+2}+\frac{1}{x+3}$ **2** $\frac{1}{x-1}+\frac{1}{x+3}$

3 $\frac{3}{x+4}-\frac{2}{x+3}$ **4** $\frac{5}{x+2}-\frac{4}{x+3}$

5 $\frac{1}{x-1}+\frac{2}{2x-1}-\frac{1}{x+3}$

6 $\frac{x+1}{x^2+4}+\frac{2}{x-2}$ **7** $\frac{1}{x-3}-\frac{2}{x^2+3}$

8 $\frac{2}{x^2+5}-\frac{2}{2x+3}$ **9** $\frac{1}{5+2x^2}-\frac{3}{x+3}$

0 $\frac{2}{2x+1}-\frac{2x+1}{x^2+2x+7}$

1 $\frac{2}{x-5}+\frac{3}{(x-5)^2}$

2 $\frac{1}{x+3}-\frac{2}{(x+3)^2}+\frac{4}{(x+3)^2}$

3 $\frac{4}{(x-1)^2}-\frac{3}{x+2}$

14 $\frac{4}{(x-1)^2}-\frac{2}{x-1}-\frac{3}{x+2}$

15 $\frac{1}{2x+1}-\frac{1}{2x+3}+\frac{3}{(2x+3)^2}$

16 $1+\frac{1}{x-1}$ **17** $x+1\frac{1}{x-1}$

18 $1+\frac{1}{x-1}-\frac{1}{x+1}$ **19** $1-\frac{2}{x}+\frac{3}{x-1}$

20 $x+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$

21 $2+\frac{3}{x+1}+\frac{1}{x-2}$

22 $2-\frac{1}{1-x}-\frac{3}{1+2x}$

23 $2x+\frac{1}{x+2}-\frac{1}{x+3}$

24 $x+1\frac{3}{x-2}-\frac{1}{x+2}$

25 $-x+\frac{1}{x}-\frac{2}{x^2}+\frac{1}{x+1}$

26 $\frac{2}{2x-3}-\frac{3}{3x+2}$

27 $\frac{2}{2x-1}+\frac{1}{x+2}-\frac{3}{(x+2)^2}$

28 $\frac{1}{x^2+1}+\frac{1}{x+1}+\frac{2}{(x+1)^2}$

29 $2x+\frac{2}{2x+3}-\frac{1}{x^2}+\frac{1}{x}$

30 $\frac{2}{2x-1}-\frac{1}{3x+1}$

31 $\frac{3}{x-1}-\frac{1}{(x-1)^2}+\frac{3-2x}{x^2+2}$

32 $\frac{11}{7(x-2)}-\frac{1}{x+2}+\frac{13}{7(2x+3)}$

33 $1-\frac{1}{x}-\frac{x}{x^2+x+1}$

34 $\frac{3}{x^2}-\frac{1}{x}+\frac{2}{x+1}$

35 $\frac{1}{x+1}+\frac{-x+2}{x^2-x+1}$

Summation of Series

- P3 book P15 Ex2A Q1,3,4,7,9,10

Exercise 2A

In each case, use the identity given to find the sum to n terms of the given series.

	Identity	Series
1	$\frac{1}{r(r+1)} \equiv \frac{1}{r} - \frac{1}{r+1}$	$\sum_{r=1}^{n} \frac{1}{r(r+1)}$
2	$2r+1 \equiv (r+1)^2 - r^2$	$\sum_{r=1}^{n} (2r+1)$
3	$\frac{2}{4r^2-1} \equiv \frac{1}{2r-1} - \frac{1}{2r+1}$	$\sum_{r=1}^{n} \frac{1}{4r^2-1}$
4	$r^2(r+1) - (r-1)^2(r) \equiv 3r^2 - r$	$\sum_{r=1}^{n} r(3r-1)$
5	$\frac{r}{r+1} - \frac{r-1}{r} \equiv \frac{1}{r(r+1)}$	$\sum_{r=1}^{n} \frac{1}{r(r+1)}$
6	$4r(r+1)(r+2) \equiv r(r+1)(r+2)(r+3) - (r-1)(r)(r+1)(r+2)$	$\sum_{r=1}^{n} r(r+1)(r+2)$
7	$\frac{2}{r(r+1)(r+2)} \equiv \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$	$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$
8	$\frac{2r+1}{r^2(r+1)^2} \equiv \frac{1}{r^2} - \frac{1}{(r+1)^2}$	$\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2}$

9 Use the identity $(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1$ to find

$$\sum_{r=1}^{n} r(r+1).$$

10 Show that $\frac{1}{r!} - \frac{1}{(r+1)!} \equiv \frac{r}{(r+1)!}$. Hence find $\sum_{r=1}^{n} \frac{r}{(r+1)!}$.

- P3 book P18 Ex2B Q1-3, 5-7, 9-11, 13, 14, 17, 19

Exercise 2B

Evaluate:

1 $\sum_{r=1}^{13} r^2$

2 $\sum_{r=4}^{11} r^3$

3 $\sum_{r=11}^{24} r(r+1)$

4 $\sum_{r=1}^{19} r(r+4)$

5 $\sum_{r=1}^{20} \frac{1}{r(r+1)}$

6 $\sum_{r=3}^{16} (r+2)^3$

7 $\sum_{r=1}^{14} \left(\frac{3}{4}\right)^r$

8 $\sum_{r=1}^{20} \frac{1}{(r+3)(r+6)}$

9 $\sum_{r=4}^{16} (2r-1)^3$

10 $\sum_{r=3}^{23} r(r+1)(r+2)$

11 Show that $\sum_{r=1}^{n} (2r-1)^2 \equiv \frac{1}{3}n(4n^2-1)$.

12 Show that $\sum_{r=1}^{n} r(2+r) \equiv \frac{1}{6}n(n+1)(2n+7)$.

13 Find $\sum_{r=1}^{20} \frac{1}{4r^2-1}$.

14 Find $\sum_{r=n}^{2n} r^2$.

15 Given that $f(r) \equiv \frac{1}{r(r+1)}$, show that

$$f(r) - f(r+1) \equiv \frac{2}{r(r+1)(r+2)}$$

Hence find $\sum_{r=5}^{25} \frac{1}{r(r+1)(r+2)}$.

16 Prove that $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$.

17 Find the sum of all even numbers between 2 and 200 inclusive, excluding those which are multiples of 3.

18 Find $\sum_{r=1}^{100} 2r^2 - \sum_{r=1}^{200} r^2$.

19 Find the sum of the series

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots - (2n)^2$$

20 Given that $u_r = r(2r+1) + 2^{r+2}$, find $\sum_{r=1}^{n} u_r$.

ANSWERS

Exercise 2A

1 $1-\dfrac{1}{n+1}$ **2** n^2+2n

3 $\dfrac{n}{2n+1}$ **4** $n^2(n+1)$

5 $\dfrac{n}{n+1}$

6 $\frac{1}{4}n(n+1)(n+2)(n+3)$

7 $\dfrac{n(n+3)}{4(n+1)(n+2)}$ **8** $\dfrac{n(n+2)}{(n+1)^2}$

9 $\frac{1}{3}n(n+1)(n+2)$ **10** $1-\dfrac{1}{(n+1)!}$

Exercise 2B

1 819 **2** 4320 **3** 4760

4 3230 **5** $\frac{20}{21}$ **6** 29 141

7 2.95 (3 s.f.) **8** 0.1655 (4 d.p.)

9 130 663 **10** 89 670 **13** $\frac{20}{41}$

14 $\dfrac{n}{6}(n+1)(14n+1)$ **15** $\frac{28}{1755}$

17 6734 **18** $-2\,010\,000$

19 $-n(2n+1)$

20 $\dfrac{n}{6}(n+1)(4n+5)+2^{n+3}-8$

Induction

- P4 book P279 Ex8A Q1-6, 9, 17, 20, 25, 29, 30, 34 Extra Q8, 13,15

Exercise 8A

In questions 1–15, use the method of mathematical induction to prove the result given.

1 $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

2 $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

3 $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

4 $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$

5 $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$

6 $1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}$

7 $\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$

8 $\sum_{r=1}^{n} \sin^2(2r-1)\theta = \frac{1}{2}n - \frac{\sin 4n\theta}{4\sin 2\theta}$

9 $\sum_{r=1}^{n} \frac{3^r(r+1)}{(r+4)!} = \frac{1}{8} - \frac{3^{n+1}}{(n+4)!}$

10 $\sum_{r=1}^{n} r(r+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5)$

11 $\sum_{r=1}^{n} \frac{2r-1}{2^{r-1}} = 6 - \frac{2n+3}{2^{n-1}}$

12 $\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$

13 $\sum_{r=1}^{n} \cos(2r-1)\theta = \frac{\sin n\theta \cos n\theta}{\sin\theta}$

14 $\sum_{r=1}^{n} \operatorname{cosec}(2^r\theta) = \cot\theta - \cot(2^n\theta)$

15 $\sum_{r=1}^{n} \tan r\theta \tan(r+1)\theta = \tan(n+1)\theta \cot\theta - n - 1$

16 Given that n is a positive integer, prove that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

17 Given that ${}^nC_r = \frac{n!}{r!\,(n-r)!}$, show that

$${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$$

Use the method of mathematical induction to prove that

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$$

where n is a positive integer.

18 Use the method of induction to show that

$$\sum_{r=1}^{2n} r^3 = n^2(2n+1)^2$$

19 Given that n is a positive integer, prove that $n(n+1)(2n+1)$ is divisible by 6.

20 Given that n is a positive integer, prove that $3^{4n+2} + 2^{6n+3}$ is divisible by 17.

21 (a) If n is an odd positive integer, prove that $2^n + 1$ is divisible by 3.

(b) If n is an even positive integer, prove that $2^n - 1$ is divisible by 3.

22 Given that n is an even positive integer, prove that $(2^{2n} - 1)$ is divisible by 5.

23 Given that n is an odd positive integer, prove that $(5^{2n} + 1)$ is divisible by 13.

24 Given that $A_n = 2^{n+2} + 3^{2n+1}$, show that

$$A_{n+1} - 2A_n = 7(3^{2n+1})$$

Hence use the method of mathematical induction to prove that A_n is divisible by 7, where n is any positive integer.

25 Given that m is an odd positive integer, prove that $(m^2 + 3)(m^2 + 15)$ is divisible by 32 for all such values of m.

26 Given that n is a positive integer, prove that $3^{2n} + 11$ is divisible by 4.

27 Given that n is a positive integer, prove that $(3n + 1)7^n - 1$ is divisible by 9.

28 Given that $0 < x < \frac{\pi}{2}$, and n is a positive integer, prove that $(1 - \sin x)^n < 1$.

29 Given that n is a positive integer, use the method of mathematical induction to prove that

$$\sum_{r=1}^{n} r^2 \geqslant n\left(\frac{n+1}{2}\right)^2$$

30 Given that n is a positive integer, prove by induction that

$$1 + 2 + 3 + \ldots + n > \tfrac{1}{2}n^2$$

31 Given that n is a positive integer, prove that

$$\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{5}{6}+\ldots+\frac{2n-1}{2n}<n-\tfrac{1}{2}, \text{ for } n\geqslant 2$$

32 Given that n is a positive integer, prove that

$$\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{5}{6}+\ldots+\frac{2n-1}{2n}>\tfrac{1}{2}n, \text{ for } n\geqslant 2$$

33 Given that n is a positive integer, show by the method of induction that

$$\frac{n}{2}<\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}+\ldots+\frac{n}{n+1}<n, \text{ for } n\geqslant 2$$

34 Given that n is a positive integer where $n\geqslant 2$, prove by the method of mathematical induction that

(a) $\displaystyle\sum_{r=1}^{n-1} r^3<\frac{n^4}{4}$

(b) $\displaystyle\sum_{r=1}^{n} r^3>\frac{n^4}{4}$

35 Given that n is a positive integer, prove by induction that

$$\sum_{r=1}^{n} r^4=\tfrac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

Maclaurin's Theorem

- P3 book P22 Ex2C Q17, 20-25

Exercise 2C

In questions 1–15, find, in ascending powers of x, the expansions up to and including the term in x^3, simplifying the coefficients. State the set of values of x for which the expansion is valid.

1 $(1+x)^{-2}$ **2** $(1-x)^{-3}$ **3** $(1-x)^{-5}$

4 $(1+x)^{-\frac{1}{2}}$ **5** $(1+x)^{\frac{3}{2}}$ **6** $(1-x)^{\frac{3}{4}}$

7 $(1-3x)^{\frac{1}{3}}$ **8** $(1+3x)^{-\frac{1}{3}}$ **9** $\left(1-\frac{1}{2}x\right)^{-2}$

10 $(1+6x)^{-1}$ **11** $(3+x)^{-1}$ **12** $(2-x)^{-2}$

13 $(4+3x)^{\frac{1}{2}}$ **14** $(8-5x)^{\frac{1}{3}}$ **15** $(100+x)^{-\frac{1}{2}}$

By using partial fractions find, in ascending powers of x, up to and including the term in x^3, expansions for the functions of x given in questions 16–20. State the set of values of x for which the expansion is valid.

16 $\dfrac{2-3x}{1-3x+2x^2}$ **17** $\dfrac{3}{1+x-2x^2}$ **18** $\dfrac{2}{x^2+2x-8}$

19 $\dfrac{1}{x^2+3x+2}$ **20** $\dfrac{8-x}{x^2-x-6}$

21 Given that $|x| < \frac{1}{2}$, expand $(1+x)^2(1-2x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient.

22 Given that $|x| > 2$ find the first four terms in the series expansion of $\left(1-\dfrac{2}{x}\right)^{\frac{1}{2}}$ in descending powers of x.

By taking $x = 200$ use your series to find a value of $\sqrt{99}$, giving your answer to 7 decimal places. Use your series to find $\sqrt{101}$ to the same degree of accuracy.

23 The series expansion of $(1+px)^q$ in ascending powers of x has coefficients of -10 and 75 in the x and x^2 terms respectively.

(a) Find the value of p and of q.

(b) Find the coefficients of the x^3 and x^4 terms in the expansion.

(c) State the set of values of x for which the series is valid.

24 Given that $|x| < 1$, expand $\left(\frac{1+x}{1-x}\right)^{\frac{1}{3}}$ in ascending powers of x up to and including the term in x^2.

25 The coefficients of x and x^2 in the expansion of $(1+px+qx^2)^{-2}$ in ascending powers of x are 4 and 14 respectively. Find the value of p and of q.

26 The coefficients of the x and x^2 terms in the expansion of $(1+px)^q$ in ascending powers of x are -6 and 6 respectively.

(a) Find the value of p and of q.

(b) Find the x^3 term and the x^4 term in the expansion.

(c) State the set of values of x for which the expansion is valid.

- P3 book P26 Ex2D

Exercise 2D

Using Maclaurin's expansion, and differentiation, show that:

1 $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^r \frac{x^r}{r!} + \cdots$

2 $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$

3 $e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots + \frac{2^r x^r}{r!} + \cdots$

4 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots$

5 $\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^r}{r} - \cdots$

Find the first three non-zero terms in the Maclaurin expansion of the function given in ascending powers of x:

6 $\tan x$

7 $\sin^2 x$

8 $\ln\left(\frac{1+x}{1-x}\right)$, $|x| < 1$

9 $(1 - 2x^2)^{\frac{1}{2}}$

10 $e^x \cos x$

Exercise 2E

1 Given that x is small, find the constants A and B such that

$$(x+\sin x)\cos x \approx Ax + Bx^3$$

2 Given that x is small, find the constants C and D such that

$$\tan x \approx Cx + Dx^3$$

3 Find $\lim\limits_{x\to 0}\left(\dfrac{\sin(\frac{\pi}{6}+x)-\sin\frac{\pi}{6}}{\sin 2x}\right)$.

4 Given that x is small, show that

$$\frac{\sin x - x\cos x}{x^3} \approx \tfrac{1}{3}$$

5 Given that x is so small that terms in x^3 and higher powers of x may be disregarded, show that

$$\ln\left[\frac{(1+2x)^2}{1-3x}\right] = 7x + \tfrac{1}{2}x^2$$

6 Show that for small x:

$$\frac{(1+x)^{\frac{1}{2}}}{(1-x)^2} \approx 1 + \tfrac{5}{2}x + \tfrac{31}{8}x^2$$

7 Given that x takes a value near $\frac{\pi}{2}$, explain why $\cos x \approx \frac{\pi}{2} - x$. Use this approximation to find (to 2 decimal places) the smallest positive root of the equation

$$\cos x = \frac{x}{10}$$

8 Given that x is small, show that

$$e^{\sin x} = 1 + x + \tfrac{1}{2}x^2 + Ax^3$$

and determine the value of A. You may assume that terms in x^4 and higher powers of x can be disregarded.

9 Evaluate: (a) $\lim_{x \to 0}\left(\dfrac{\sin x - x}{\sin x - x\cos x}\right)$ (b) $\lim_{x \to 0}\left(\dfrac{\ln(1+x) - x}{\sin^2 x}\right)$.

10 Given that x is small and that terms in x^4 and higher powers of x may be disregarded, show that

$$\ln(\sec x + \tan x) = x + \tfrac{1}{6}x^3$$

ANSWERS

Exercise 2C

1 $1 - 2x + 3x^2 - 4x^3 + \dots,$ $|x| < 1$

2 $1 + 3x + 6x^2 + 10x^3 + \dots,$ $|x| < 1$

3 $1 + 5x + 15x^2 + 35x^3 + \dots,$ $|x| < 1$

4 $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots,$ $|x| < 1$

5 $1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 \dots,$ $|x| < 1$

6 $1 - \frac{3}{4}x - \frac{3}{32}x^2 - \frac{5}{128}x^3 \dots$ $|x| < 1$

7 $1 - x - x^2 - \frac{5}{3}x^3 - \dots$ $|x| < \frac{1}{3}$

8 $1 - x + 2x^2 - \frac{14}{3}x^3 + \dots$ $|x| < \frac{1}{3}$

9 $1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \dots$ $|x| < 2$

10 $1 - 6x + 36x^2 - 216x^3 + \dots$ $|x| < \frac{1}{6}$

11 $\frac{1}{3} - \frac{x}{9} + \frac{x^2}{27} - \frac{x^3}{81} + \dots$ $|x| < 3$

12 $\frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \dots$ $|x| < 2$

13 $2 + \frac{3}{4}x - \frac{9}{64}x^2 + \frac{27}{512}x^3 - \dots$ $|x| < \frac{4}{3}$

14 $2 - \frac{5}{12}x - \frac{25}{288}x^2 - \frac{625}{20736}x^3 - \dots$ $|x| < \frac{8}{5}$

15 $\frac{1}{10} - \frac{1}{2000}x + \frac{3}{800\,000}x^2 - \frac{1}{32\,000\,000}x^3 + \dots$ $|x| < 100$

16 $2 + 3x + 5x^2 + 9x^3 + \dots$ $|x| < \frac{1}{2}$

17 $3 - 3x + 9x^2 - 15x^3 + \dots$ $|x| < \frac{1}{2}$

18 $-\frac{1}{4} - \frac{x}{16} - \frac{3}{64}x^2 - \frac{5}{256}x^3 - \dots$ $|x| < 2$

19 $\frac{1}{2} - \frac{3}{4}x + \frac{7}{8}x^2 - \frac{15}{16}x^3 + \dots$ $|x| < 1$

20 $-\frac{4}{3} + \frac{7}{18}x - \frac{31}{108}x^2 + \frac{73}{648}x^3 - \dots$ $|x| < 2$

21 $1 + 3x + \frac{9}{2}x^2 + \frac{13}{2}x^3 + \dots$

22 $1 - x^{-1} - \frac{1}{2}x^{-2} - \frac{1}{2}x^{-3}$, 9.949 874 4; 10.049 875 6

23 (a) $p = 5, q = -2$ (b) $-500, 3125$ (c) $|x| < \frac{1}{5}$

24 $1 + \frac{2}{3}x + \frac{2}{9}x^2$

25 $p = -2, q = -1$

26 (a) $p = -4, q = \frac{3}{2}$ (b) $4x^3, 6x^4$ (c) $|x| < \frac{1}{4}$

Exercise 2D

6 $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \dots$

7 $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 \dots$

8 $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 \dots$

9 $1 - x^2 - \frac{1}{2}x^4 \dots$

10 $1 + x - \frac{1}{3}x^3 \dots$

Exercise 2E

1 $A = 2, B = -\frac{7}{6}$ 2 $C = 1, D = \frac{1}{3}$

3 $\frac{1}{4}\sqrt{3}$ 7 1.43 8 $A = 0$

9 (a) $-\frac{1}{2}$ (b) $-\frac{1}{2}$

Section 1: Improper integrals

Section test

1. Which of the following integrals are improper integrals?

   (i) $\int_0^\infty x\,\mathrm{d}x$ (ii) $\int_0^1 \frac{1}{x}\,\mathrm{d}x$

   (iii) $\int_0^1 \frac{1}{x+1}\,\mathrm{d}x$ (iv) $\int_1^3 \frac{1}{x-2}\,\mathrm{d}x$

2. The value of the integral $\int_1^\infty \frac{1}{x^{1/3}}\,\mathrm{d}x$ is

(a) $\frac{3}{2}$
(b) 3
(c) $-\frac{3}{2}$
(d) undefined

3. The value of the integral $\int_0^8 \frac{1}{x^{1/3}}\,\mathrm{d}x$ is

(a) $\frac{3}{2}$
(b) 3
(c) 6
(d) undefined

4. The value of the integral $\int_1^\infty \frac{1}{x^3}\,\mathrm{d}x$ is

(a) $\frac{1}{2}$
(b) 1
(c) $-\frac{1}{2}$
(d) undefined

5. The value of the integral $\int_0^2 \frac{1}{x^3}\,\mathrm{d}x$ is

(a) $\frac{1}{2}$
(b) $-\frac{1}{8}$
(c) $\frac{1}{8}$
(d) undefined

6. Which of the following integrals can be evaluated?

   (i) $\int_{-2}^\infty \frac{1}{x^2}\,\mathrm{d}x$

   (ii) $\int_0^\infty \frac{1}{\sqrt{x}}\,\mathrm{d}x$

(a) (i) only
(b) (ii) only
(c) both
(d) neither

CCEA FM Further calculus 1 section test solutions

Solutions to section test

1. (i) is an improper integral as one of the limits is infinity
   (ii) is an improper integral as the integrand is undefined at $x = 0$
   (iii) is not an improper integral (the integrand is undefined at $x = -1$, but this is not between the limits of the integral
   (iv) is an improper integral as the integrand is undefined at $x = 2$.

2. $$\int_1^a \frac{1}{x^{1/3}}dx = \int_1^a x^{-1/3}dx$$
$$= \left[\tfrac{3}{2}x^{2/3}\right]_1^a$$
$$= \tfrac{3}{2}a^{2/3} - \tfrac{3}{2}$$
As $a \to \infty$, $a^{2/3} \to \infty$, so the integral is undefined.

3. $$\int_a^8 \frac{1}{x^{1/3}}dx = \int_a^8 x^{-1/3}dx$$
$$= \left[\tfrac{3}{2}x^{2/3}\right]_a^8$$
$$= \tfrac{3}{2}\times 8^{2/3} - \tfrac{3}{2}a^{2/3}$$
$$= 6 - \tfrac{3}{2}a^{2/3}$$
As $a \to 0$, $a^{2/3} \to 0$, so the value of the integral is 6.

4. $$\int_1^a \frac{1}{x^3}dx = \int_1^a x^{-3}\,dx$$
$$= \left[-\tfrac{1}{2}x^{-2}\right]_1^a$$
$$= -\frac{1}{2a^2} + \frac{1}{2}$$
As $a \to \infty$, $\frac{1}{2a^2} \to 0$, so the value of the integral is $\frac{1}{2}$.

5. $$\int_a^2 \frac{1}{x^3}dx = \int_a^2 x^{-3}\,dx$$
$$= \left[-\tfrac{1}{2}x^{-2}\right]_a^2$$
$$= -\frac{1}{8} + \frac{1}{2a^2}$$
As $a \to 0$, $\frac{1}{2a^2}$ is undefined, so the integral is undefined.

6. (i) $\int_{-2}^{a} \frac{1}{x^2} dx = \int_{-2}^{a} x^{-2} dx + \int_{b}^{c} x^{-2} dx$

$= \left[-x^{-1}\right]_{-2}^{a} + \left[-x^{-1}\right]_{b}^{c}$

$= -\frac{1}{a} - \frac{1}{2} - \frac{1}{c} + \frac{1}{b}$

As $a \to 0$, $b \to 0$ and $c \to \infty$, $\frac{1}{a}$ and $\frac{1}{b} \to 0$, and $\frac{1}{c}$ is undefined, so the integral is undefined.

(ii) $\int_{0}^{a} \frac{1}{\sqrt{x}} dx = \int_{0}^{a} x^{-1/2} dx$

$= \left[2x^{1/2}\right]_{0}^{a}$

$= 2\sqrt{a}$

As $a \to \infty$, $\sqrt{a}$ is undefined, so the integral is undefined.

Differentiation of arcsin, arcos, arctan

-P3 book P52 Ex3C Q7-15

Exercise 3C

1 Find, in radians in terms of π, the value of:

(a) $\arcsin 1$ (b) $\arcsin(-\frac{\sqrt{3}}{2})$ (c) $\arccos \frac{\sqrt{3}}{2}$

(d) $\arccos 0$ (e) $\arctan(-\sqrt{3})$ (f) $\arctan(2+\sqrt{3})$

2 Giving your answer in radians to 2 decimal places, find the value of:

(a) $\arcsin(0.75)$ (b) $\arctan 7$ (c) $\arccos(-0.735)$

(d) $\arcsin(-0.993)$ (e) $\arccos(-0.111)$ (f) $\arctan(-0.352)$

3 Given that $y = \operatorname{cosec} x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, $x \neq 0$, sketch the graphs of the curves $y = \operatorname{cosec} x$ and $y = \operatorname{arccosec} x$, where $\operatorname{arccosec} x$ is the inverse function of $\operatorname{cosec} x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, $x \neq 0$.

4 Given that $y = \cot x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $x \neq 0$, sketch the graphs of the curves $y = \cot x$ and $y = \operatorname{arccot} x$, where $\operatorname{arccot} x$ is the inverse function of $\cot x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $x \neq 0$.

5 Given that $y = \sec x$, $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$, sketch the graphs of the curves $y = \sec x$ and $y = \operatorname{arcsec} x$, where $\operatorname{arcsec} x$ is the inverse function of $\sec x$, $0 \leqslant x \leqslant \pi$, $x \neq \frac{\pi}{2}$.

6 Find the smallest positive value of x for which

(a) $\tan 2x = \sqrt{3}$

(b) $\sin(2x-3) = \frac{1}{2}$

(c) $\sin x = \cos(\arctan 1)$

7 Differentiate with respect to x:

(a) $\arcsin 3x$ (b) $(\arcsin x)^2$ (c) $\arcsin\left(\frac{1}{x}\right)$

8 Differentiate with respect to x:

(a) $\arccos\left(\frac{x}{2}\right)$ (b) $\arccos(3x^2)$ (c) $\arccos\left(\frac{1}{x+2}\right)$

9 Differentiate with respect to x:

(a) $\arctan(2x)$ (b) $e^{\arctan x}$ (c) $\arctan(\ln x)$

10 Differentiate with respect to x:

(a) $x \arcsin x$ (b) $e^x \arccos x$ (c) $\dfrac{e^x}{\arctan x}$

(d) $\arctan\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)$ (e) $\operatorname{arcsec} x^2$

11 Given that $y = \arcsin x$, show that:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

12 Given that $y = x - \arctan x$, show that:

$$\frac{d^2y}{dx^2} - 2x\left(1 - \frac{dy}{dx}\right)^2 = 0$$

13 Find (a) $\frac{d}{dx}(\text{arcsec}\, x)$ (b) $\frac{d}{dx}(\text{arccot}\, x)$.

14 Find an equation at the point where $x = \frac{2}{\sqrt{3}}$ of the tangent to the curve $y = \text{arccosec}\, x$.

15 Given that k is a positive constant, differentiate

(a) $\arccos \frac{x}{k}$ (b) $\arcsin \frac{k}{x}$ (c) $\arctan \frac{k}{x}$

ANSWERS

Exercise 3C

1 (a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$
(d) $\frac{\pi}{2}$ (e) $-\frac{\pi}{3}$ (f) $\frac{5\pi}{12}$

2 (a) 0.85 (b) 1.43 (c) 2.40
(d) −1.45 (e) 1.68 (f) −0.34

6 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}+\frac{3}{2}$ (c) $\frac{\pi}{4}$

7 (a) $\frac{3}{\sqrt{(1-9x^2)}}$ (b) $\frac{2\arcsin x}{\sqrt{(1-x^2)}}$

(c) $\frac{-1}{x\sqrt{(x^2-1)}}$

8 (a) $\frac{-1}{\sqrt{(4-x^2)}}$ (b) $\frac{-6x}{\sqrt{(1-9x^4)}}$

(c) $\frac{-1}{(x+2)\sqrt{(x^2+4x+3)}}$

9 (a) $\frac{2}{1+4x^2}$ (b) $\frac{e^{\arctan x}}{1+x^2}$

(c) $\frac{1}{x[1+(\ln x)^2]}$

10 (a) $\arcsin x + \frac{x}{\sqrt{(1-x^2)}}$

(b) $e^x \arccos x - \frac{e^x}{\sqrt{(1-x^2)}}$

(c) $\frac{e^x \arctan x - \frac{e^x}{1+x^2}}{(\arctan x)^2}$

(d) $\frac{-1}{2(1+x)\sqrt{x}}$ (e) $\frac{2}{x\sqrt{(x^4-1)}}$

13 (a) $\frac{1}{x\sqrt{(x^2-1)}}$ (b) $\frac{-1}{1+x^2}$

14 $y - \frac{\pi}{3} = -\frac{3}{2}\left(x - \frac{2}{\sqrt{3}}\right)$

15 (a) $\frac{-1}{\sqrt{(k^2-x^2)}}$ (b) $\frac{-k}{x\sqrt{(x^2-k^2)}}$

(c) $\frac{-k}{x^2+k^2}$

Integration of $\frac{1}{a^2+x^2}$ and $\frac{1}{\sqrt{a^2-x^2}}$

-UPM P376 Ex15H Q13-24

Exercise 15H

Differentiate the following with respect to x

1. $\sin^{-1} x$
2. $\tan^{-1} \frac{x}{a}$
3. $\sin^{-1} \frac{x}{4}$
4. $\cos^{-1} 3x$
5. $\tan^{-1} 4x$
6. $\sin^{-1} 6x$
7. $\sin^{-1} (2x - 1)$
8. $\tan^{-1} (1 - 3x)$
9. $\sin^{-1} (x^2 - 1)$
10. $x \sin^{-1} x$
11. $x \tan^{-1} x$
12. $(x^2 + 1) \tan^{-1} x$

Find the following indefinite integrals

13. $\int \frac{1}{\sqrt{(4 - x^2)}} dx$
14. $\int \frac{1}{\sqrt{(16 - x^2)}} dx$
15. $\int \frac{3}{9 + x^2} dx$
16. $\int \frac{1}{25 + x^2} dx$
17. $\int \frac{1}{\sqrt{(49 - x^2)}} dx$
18. $\int \frac{1}{49 + x^2} dx$
19. $\int \frac{1}{\sqrt{(25 - x^2)}} dx$
20. $\int \frac{2}{100 + x^2} dx$

Evaluate the following definite integrals (leave π in your answers).

21. $\int_{-3}^{3} \frac{1}{\sqrt{(36 - x^2)}} dx$
22. $\int_{-2}^{2} \frac{1}{4 + x^2} dx$
23. $\int_{-\sqrt{3}}^{3} \frac{\sqrt{3}}{x^2 + 3} dx$
24. $\int_{0}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{(3 - x^2)}} dx$

ANSWERS

page 376 ***Exercise 15H***

1. $\frac{1}{\sqrt{(1 - x^2)}}$
2. $\frac{a}{a^2 + x^2}$
3. $\frac{1}{\sqrt{(16 - x^2)}}$
4. $\frac{-3}{\sqrt{(1 - 9x^2)}}$
5. $\frac{4}{1 + 16x^2}$
6. $\frac{6}{\sqrt{(1 - 36x^2)}}$
7. $\frac{1}{\sqrt{(x(1 - x))}}$
8. $\frac{-3}{2 - 6x + 9x^2}$
9. $\frac{2}{\sqrt{(2 - x^2)}}$
10. $\sin^{-1} x + \frac{x}{\sqrt{(1 - x^2)}}$
11. $\tan^{-1} x + \frac{x}{1 + x^2}$
12. $2x \tan^{-1} x + 1$
13. $\sin^{-1}\left(\frac{x}{2}\right) + c$
14. $\sin^{-1}\left(\frac{x}{4}\right) + c$
15. $\tan^{-1}\left(\frac{x}{3}\right) + c$
16. $\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c$
17. $\sin^{-1}\left(\frac{x}{7}\right) + c$
18. $\frac{1}{7}\tan^{-1}\left(\frac{x}{7}\right) + c$
19. $\sin^{-1}\left(\frac{x}{5}\right) + c$
20. $\frac{1}{5}\tan^{-1}\left(\frac{x}{10}\right) + c$
21. $\frac{\pi}{3}$
22. $\frac{\pi}{4}$
23. $\frac{7}{12}\pi$
24. $\frac{\pi}{3}$

Repeated integration by parts/reduction formula

-P2 book P289 Ex9D Q1,3,5,6,9, 14-16,18,19

Exercise 9D

Use integration by parts to find:

1 $\int xe^{-x}\,dx$ **2** $\int xe^{3x}\,dx$ **3** $\int x\sin x\,dx$

4 $\int x\ln x\,dx$ **5** $\int \ln(x-1)\,dx$ **6** $\int x\cos 3x\,dx$

7 $\int x(x-1)^4\,dx$ **8** $\int x\sqrt{(x-1)}\,dx$ **9** $\int x^2e^x\,dx$

10 $\int x^2\cos x\,dx$ **11** $\int x^2e^{-x}\,dx$ **12** $\int x^3\ln x\,dx$

Evaluate each of the following definite integrals:

13 $\int_0^{\pi} x\sin x\,dx$ **14** $\int_0^{\frac{\pi}{2}} x\cos\frac{1}{2}x\,dx$ **15** $\int_1^{e} x^2\ln x\,dx$

16 $\int_0^{1} x(x-1)^3\,dx$ **17** $\int_0^{2} (x-1)(x+1)^3\,dx$

18 $\int_1^{e} \frac{\ln x}{x^4}\,dx$ **19** $\int_1^{e} (\ln x)^2\,dx$ **20** $\int_0^{\frac{\pi}{2}} e^x\sin x\,dx$

ANSWERS

Exercise 9D

[The constant of integration is omitted in indefinite integration.]

1 $-e^{-x}(x+1)$ **2** $\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}$

3 $-x\cos x+\sin x$

4 $\frac{x^2}{2}\ln|x|-\frac{x^2}{4}$

5 $x\ln|x-1|-x-\ln|x-1|$

6 $\frac{1}{3}x\sin 3x+\frac{1}{9}\cos 3x$

7 $\frac{(5x+1)(x-1))^5}{30}$ **8** $\frac{2}{15}(3x+2)(x-1)^{\frac{3}{2}}$

9 $e^x(x^2-2x+2)$

10 $x^2\sin x+2x\cos x-2\sin x$

11 $-e^{-x}(x^2+2x+2)$

12 $\frac{x^4}{16}(4\ln x-1)$ **13** π

14 $\frac{\pi}{\sqrt{2}}+\frac{4}{\sqrt{2}}-4$ **15** $\frac{2}{9}e^3+\frac{1}{9}$ **16** $-\frac{1}{20}$

17 8.4 **18** $\frac{1}{9}(1-4e^{-3})$ **19** $e-2$

20 $\frac{1}{2}(e^{\frac{\pi}{2}}+1)$

-P4 book page 202 Ex5A Q 1,2,3,5,7,9-13 leave Q4 and 6 to the end

Exercise 5A

1 Given that $I_n = \int \cos^n x \, dx$, show that, for $n \geqslant 2$,

$$nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$$

Hence evaluate I_8 for $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$, and I_9 for $\int_0^{\pi} \cos^n x \, dx$.

2 Given that $I_n = \int (\ln x)^n dx$, show that, for $n \geqslant 1$,

$$I_n = x(\ln x)^n - nI_{n-1}$$

Hence evaluate $\int_1^2 (\ln x)^3 dx$.

3 Given that $I_n = \int_0^{2\pi} \sin^n x \, dx$, find a reduction relation between I_n and I_{n-2} for $n \geqslant 2$. Hence find I_5 and I_6.

4 Evaluate (a) $\int_0^{\frac{\pi}{2}} \sin^3 \theta \cos^4 \theta \, d\theta$ (b) $\int_0^{\frac{\pi}{2}} \sin^2 \theta \cos^6 \theta \, d\theta$.

5 Use the substitution $x = \sin t$ and an appropriate reduction formula to evaluate

$$\int_0^1 x^6 (1-x^2)^{\frac{1}{2}} \, dx$$

6 Use the substitution $x = \sin^2 t$ and an appropriate reduction formula to evaluate

$$\int_0^1 x(1-x)^{\frac{3}{2}} \, dx$$

Check your answer by using another method.

7 Using an appropriate reduction formula, evaluate, in terms of e, the integral $\int_0^1 x^5 e^x \, dx$.

8 Given that $I_n = \int \cosh^n x \, dx$, show that, for $n \geqslant 2$,

$$nI_n = \cosh^{n-1} x \sinh x + (n-1)I_{n-2}$$

Hence find $\int \cosh^4 x \, dx$.

9 Given that $I_n = \int_0^1 x^n e^{-x} dx$, show that, for $n \geqslant 1$,

$$I_n = nI_{n-1} - e^{-1}$$

Hence evaluate $\int_0^1 x^6 e^{-x} dx$.

10 Given that $I_n = \int_0^{\frac{\pi}{3}} \sec^n x \, dx$ show that, for $n \geqslant 2$,

$$I_n = \frac{2^{n-2}}{n-1}\sqrt{3} + \frac{n-2}{n-1} I_{n-2}$$

Hence evaluate I_7.

11 Given that $I_n = \int_0^1 \frac{1}{(1+x^2)^n} dx$, show that, for $n \geqslant 2$,

$$2(n-1)I_n = 2^{1-n} + (2n-3)I_{n-1}$$

Hence find I_3.

12 Given that $I_n = \int \frac{\sin nx}{\sin x} dx$, $n \geqslant 2$,

show that $$I_n = \frac{2 \sin (n-1)x}{n-1} + I_{n-2}$$

Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin 4x}{\sin x} dx$ and check your answer by using another method.

13 Given that $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$, show that, for $n \geqslant 2$,

$$I_n = n\left(\tfrac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$$

Hence evaluate I_3.

14 By setting up an appropriate reduction formula for $\int x^n \sinh x \, dx$ and applying the formula show that

$$\int x^5 \sinh x \, dx = \cosh x\,(x^5 + 20x^3 + 120x) - \sinh x\,(5x^4 + 60x^2 + 120) + C$$

where C is a constant.

15 Given that $I_n = \int_0^2 (4 - x^2)^n \, dx$, show that, for $n \geqslant 1$,

$$I_n = \frac{8n}{2n+1} I_{n-1}$$

Evaluate I_4.

ANSWERS

Exercise 5A

1 $\frac{35}{256}\pi$, $\frac{128}{315}$

2 $2(\ln 2)^3 - 6(\ln 2)^2 + 12\ln 2 - 6$

3 $I_n = \dfrac{n-1}{n} I_{n-2}$, $I_5 = 0$, $I_6 = \frac{5}{8}\pi$

4 (a) $\frac{2}{35}$ (b) $\frac{5}{256}\pi$

5 $\frac{5}{256}\pi$

6 $\frac{4}{35}$

7 $120 - 44\mathrm{e}$

8 $\frac{1}{4}\cosh^3 x \sinh x + \frac{3}{8}\cosh x \sinh x + \frac{3}{8}x + C$

9 $720 - 1957\mathrm{e}^{-1}$

10 $\frac{61}{8}\sqrt{3} + \frac{5}{16}\ln(2 + \sqrt{3})$

11 $\frac{1}{4} + \frac{3}{32}\pi$

12 $\frac{4}{3}(1 - \sqrt{2})$

13 $\frac{3}{4}(\pi^2 - 8)$

15 $\frac{65\,536}{315}$

Hyperbolic functions, Inverse hyp fns, their definitions, derivatives and special integrals

-P3 book P76 Ex4A Q(1-3)alt,4,5,7-17odds,18,20,22,23,25

-P3 book P76 Ex4A Q26,27,29,31-33, 35,38,40

Exercise 4A

1 Express in terms of e:

(a) $\sinh 2$ (b) $\cosh \frac{1}{2}$ (c) $\tanh(-3)$

(d) $\cosh(\sqrt{2})$ (e) $\sinh \pi$ (f) $\tanh 1 - \tanh(-1)$

2 Find, to 3 decimal places, the values of x for which:

(a) $\sinh x = 3$ (b) $\sinh x = -3$ (c) $\cosh x = \frac{3}{2}$

(d) $\cosh x = \sqrt{5}$ (e) $\tanh x = \frac{3}{4}$ (f) $\tanh x = -\frac{2}{3}$

3 Find the value of each of the following, giving each answer to 4 significant figures:

(a) $\cosh 4$ (b) $\sinh \frac{2}{3}$ (c) $\tanh(-2)$

(d) $\sinh(-\frac{1}{2})$ (e) $\cosh \pi$ (f) $\tanh(e^{\frac{1}{2}})$

4 Given that $\cosh x = \frac{5}{3}$, show that $\sinh x = \pm\frac{4}{3}$. Hence find the values of e^x and x.

5 Sketch, in separate diagrams, the curves with equations

(a) $y = \operatorname{cosech} x$, $x \in \mathbb{R}$, $x \neq 0$

(b) $y = \coth x$, $x \in \mathbb{R}$, $x \neq 0$

Give the equations of the asymptotes to each curve.

6 Sketch, in the same diagram, the curves with equations $y = \sinh 2x$ and $y = \sinh 3x$.

Find the x-coordinates of the points where the curves meet the line $y = 2$, giving your answer to 2 decimal places.

In questions 7–20, prove the given identity and, where appropriate, check the identity independently by using Osborn's rule when you know the comparable trigonometric identity.

7 $\sinh A \equiv -\sinh(-A)$

8 $\sinh 2A \equiv 2\sinh A \cosh A$

9 $\cosh 2A \equiv 2\cosh^2 A - 1$

10 $\sinh 3A \equiv 3\sinh A + 4\sinh^3 A$

11 $\cosh 3A \equiv 4\cosh^3 A - 3\cosh A$

12 $\tanh^2 A + \operatorname{sech}^2 A \equiv 1$

13 $\sinh(A - B) \equiv \sinh A \cosh B - \cosh A \sinh B$

14 $\cosh(A - B) \equiv \cosh A \cosh B - \sinh A \sinh B$

15 $\cosh A + \cosh B \equiv 2\cosh\dfrac{A+B}{2}\cosh\dfrac{A-B}{2}$

16 $\sinh A + \sinh B \equiv 2\sinh\dfrac{A+B}{2}\cosh\dfrac{A-B}{2}$

17 $2\sinh A \sinh B \equiv \cosh(A+B) - \cosh(A-B)$

18 $\dfrac{\cosh x - 1}{\cosh x + 1} \equiv \tanh^2 \dfrac{x}{2}$

19 $\sinh x \equiv \dfrac{2\tanh \frac{x}{2}}{1 - \tanh^2 \frac{x}{2}}$

20 $\dfrac{\cosh x + \sinh x + 1}{\cosh x + \sinh x - 1} \equiv \coth \dfrac{x}{2}$

21 Given that $\sinh x = \tan\theta$, $0 < \theta < \frac{\pi}{2}$, express $\cosh x$ and $\tanh x$ in terms of θ.

22 Given that $x > 0$, show that

$$\sinh(\ln x) = \frac{x^2 - 1}{2x}$$

Express $\cosh(\ln x)$ in a similar form.

23 Find the value, or values, of x for which

$$4\sinh x - 3\cosh x = 5$$

giving your answer, or answers, to 3 significant figures.

24 Given that $\tanh t = \frac{1}{3}$, find the value of e^{2t}. Hence find the exact value of t.

25 Using Maclaurin's expansion for e^x and e^{-x}, express $\sinh x$ and $\cosh x$ as power series in increasing powers of x, up to and including terms in x^5 and x^6 respectively.

26 Given that $\sinh y = x$, show that

$$y = \ln[x + (1 + x^2)^{\frac{1}{2}}]$$

By differentiating this result, show that

$$(1 + x^2)\left(\frac{dy}{dx}\right)^2 = 1$$

27 Solve the equation $2\cosh x + \sinh x = 2$.

28 Solve the equation $13\cosh\theta + 12\sinh\theta = \frac{25}{4}$.

29 Prove that $\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y$.
Given that $a\cosh t + b\sinh t = R\cosh(t + \alpha)$, $a > b > 0$, show that

$$\alpha = \frac{1}{2}\ln\left(\frac{a + b}{a - b}\right)$$

Find R in terms of a and b.

30 Using the definitions of $\sinh x$ and $\cosh x$, in terms of e^x, show that for $|x| < 1$,

$$\operatorname{artanh} x = \frac{1}{2}\ln\frac{1 + x}{1 - x}$$

Hence expand $\operatorname{artanh} x$ in ascending powers of x up to and including the term in x^5.

31 Solve for x the equation

$$3\operatorname{sech}^2 x + 4\tanh x + 1 = 0$$

giving the root as a natural logarithm.

32 Solve the equation

$$\cosh^2 t + \sinh^2 t = 3$$

giving the answers in terms of natural logarithms.

33 Solve the equation

$$4\tanh t - \operatorname{sech} t = 1$$

giving the answer in terms of a natural logarithm.

34 Prove that $\operatorname{arsinh} x = \ln[x + (1 + x^2)^{\frac{1}{2}}]$.
Given that x is large and positive, show that:

$$\operatorname{arsinh} x \approx \ln 2 + \ln x + \frac{1}{4x^2}$$

35 Solve the equation $\cosh 2x = 3\sinh x$, giving your answers to 3 significant figures.

36 Given that $p = \frac{1}{2}\ln 2$, find the value of $\tanh p$. Find also the values of $\sinh 2p$, $\cosh 2p$ and $\tanh 2p$.

37 Prove that $\coth A + \operatorname{cosech} A \equiv \coth \dfrac{A}{2}$.

38 Given that $x = \sin\theta \cosh t$ and $y = \cos\theta \sinh t$, find a relation between

(a) x, y and θ (b) x, y and t.

39 Prove that $\lim\limits_{x\to 0} \dfrac{\sinh x - \sin x}{x^3} = \frac{1}{3}$

40 Prove that $\cosh^6 A - \sinh^6 A \equiv 1 + \frac{3}{4}\sinh^2 2A$.
Hence show that

$$8(\cosh^6 A - \sinh^6 A) \equiv 3\cosh 4A + 5$$

Exercise 4B

In questions 1–20, differentiate with respect to x:

1 $\cosh 2x$ **2** $\sinh \frac{x}{2}$ **3** $\tanh 3x$

4 $\operatorname{sech} 2x$ **5** $\operatorname{cosech} \frac{x}{3}$ **6** $e^x \cosh x$

7 $\sinh^2 3x$ **8** $\tanh^3 x$ **9** $\coth(\ln x)$

10 $\ln(\sinh x)$ **11** $x \sinh 2x$ **12** $x^3 \cosh 3x$

13 $\ln(\tanh x)$ **14** $e^{\sinh x}$ **15** $\frac{x}{\cosh x}$

16 $\frac{\cosh x}{x}$ **17** $e^{\cosh^3 x}$ **18** $\frac{\coth 2x}{x^3}$

19 $\frac{\operatorname{cosech}(x^2)}{x}$ **20** $\ln(\tanh x - \operatorname{sech} x)$

21 Given that $y = \operatorname{arsinh}(x - 1)$, find the value of $\frac{dy}{dx}$ at $x = 2$.

22 Find the equation of the normal at the point where $x = \ln 2$ on the curve $y = \sinh x + 3\cosh x$.

23 The curve $y = 5\sinh x - 4\cosh x$ crosses the x-axis at the point A. Determine the coordinates of A and the equation of the tangent to the curve at A.

24 Find the minimum value of y, where $y = 13\cosh x + 12\sinh x$ and the value of x where this occurs.

25 The tangent at the point P with x-coordinate $2c$ on the curve with equation $y = c\cosh \frac{x}{c}$, meets the y-axis at the point Q. Find the distance OQ in terms of c, where O is the origin.

26 Find to 2 decimal places the coordinates of the stationary points on the curve $y = 8\sinh x - 27\tanh x$ and determine the nature of these stationary points.

27 Given that $y = A\cosh 3x + B\sinh 3x$, where A and B are constants, show that $\frac{d^2y}{dx^2} - 9y = 0$.

28 Use successive differentiation and Maclaurin's expansion to show that:

- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$

29 Given that $y = \cosh 3x \sin x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

30 Find the first two non-zero terms in the series expansion of $\tanh x$ in ascending powers of x.

31 Find an equation of the tangent and an equation of the normal at the point where $x = \frac{3}{2}$ on the curve with equation $y = \tanh x$.

In questions 32–47, differentiate with respect to x:

32 $\operatorname{arsinh} x$ **33** $\operatorname{arcosh} \frac{x}{2}$ **34** $\operatorname{artanh} x^2$

35 $\operatorname{arsech} x$ **36** $\operatorname{arcosech} x$ **37** $\operatorname{arcoth} 2x$

38 arsech $x^{\frac{1}{2}}$ **39** x arcosh x **40** $\dfrac{x}{\text{arsinh}\, x}$

41 $(\text{artanh}\, x)^2$ **42** $(\text{arsech}\, x)^{\frac{1}{2}}$ **43** $e^{x^2}\,\text{arsinh}\, x$

44 $\dfrac{\ln x}{\text{arcosh}\, x}$ **45** artanh $(\sin x)$ **46** artanh$(\sinh x)$

47 $\dfrac{\arcsin x}{\text{arsinh}\, x}$

48 Find an equation of the tangent to the curve $y = \text{arsinh}\, x$ at

(i) the origin and

(ii) the point where $x = 1$.

49 Given that $y = (\text{arsinh}\, x)^2$, show that:

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$$

50 Find an equation of the normal at the point where $x = \frac{3}{4}$ on the curve with equation $y = \text{artanh}\, x$.

51 Given that $y = \text{arsinh}\, x$, show that

(a) $y = \ln[x + \sqrt{(1 + x^2)}]$

(b) $(1 + x^2)\left(\dfrac{dy}{dx}\right)^2 = 1$

(c) $(1 + x^2)\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

52 Show that the curve with equation $y = 3\cosh x - x\sinh x$ has a minimum point A on the y-axis. Find the coordinates of A. Show further that the curve has another stationary value between $x = 1.9$ and $x = 2$. Sketch the curve.

53 Show that $y = e^{\text{arsinh}\, x}$ satisfies the relation

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

54 Given that $y = \sinh x + k\cosh x$, show that the least value of y is $\sqrt{(k^2-1)}$ and that this occurs at $x = \frac{1}{2}\ln\left(\frac{k-1}{k+1}\right)$ where k is a constant and $|k| > 1$.

55 Show that $(\cosh x + \sinh x)^k + (\cosh x - \sinh x)^k \equiv 2\cosh kx$, where k is real.

Hence solve the equation

$$(\cosh x + \sinh x)^5 + (\cosh x - \sinh x)^5 = 5$$

giving your answers to 2 decimal places.

56 Find the coordinates of the minimum point on the curve $y = 5\cosh x - 3\sinh x$.

57 Given that $y = \arctan(e^x)$, show that $\frac{dy}{dx} = \frac{1}{2}\operatorname{sech} x$, and find $\frac{d^2y}{dx^2}$.

58 Given that $\operatorname{artanh} x + \operatorname{artanh} y = \frac{1}{2}\ln 5$, show that $y = \frac{2-3x}{3-2x}$.

59 Given that $y = \ln\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$, show that $\sinh y = \tan x$ and $\cosh y = \sec x$.

60 For the curve $y = \operatorname{arsinh}(x+1)$, find

(a) the coordinates of its point of inflexion P

(b) the equation of the normal to the curve at P.

ANSWERS

Exercise 4A

1 (a) $\frac{1}{2}(e^2 - e^{-2})$ (b) $\frac{1}{2}(e^{\frac{1}{2}} + e^{-\frac{1}{2}})$
(c) $-\dfrac{e^3 - e^{-3}}{e^3 + e^{-3}}$ (d) $\frac{1}{2}(e^{\sqrt{2}} + e^{-\sqrt{2}})$
(e) $\frac{1}{2}(e^{\pi} - e^{-\pi})$ (f) $2\left(\dfrac{e - e^{-1}}{e + e^{-1}}\right)$

2 (a) 1.818 (b) −1.818 (c) ±0.962
(d) ±1.444 (e) 0.973 (f) −0.805

3 (a) 27.31 (b) 0.7172 (c) −0.9640
(d) −0.5211 (e) 11.59 (f) 0.9287

4 $e^x = 3$ or $\frac{1}{3}$, $x = \pm \ln 3$

5 (a) $x = 0, y = 0$ (b) $y = \pm 1, x = 0$

6 0.72, 0.48

21 $\sec\theta, \sin\theta$ 22 $\dfrac{x^2+1}{2x}$

23 2.37

24 2, $\frac{1}{2}\ln 2$

25 $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dots$

$\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \dots$

27 $0, -\ln 3$ 28 $-\ln\frac{5}{2}, -\ln 10$

29 $\sqrt{(a^2 - b^2)}$ 30 $x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots$

31 $-\frac{1}{2}\ln 5$ 32 $\pm\frac{1}{2}\ln(3 + 2\sqrt{2})$

33 $\ln\frac{5}{3}$ 35 0.481, 0.881

36 $\frac{1}{3}; \frac{3}{4}, \frac{5}{4}, \frac{3}{5}$

38 (a) $x^2 \operatorname{cosec}^2\theta - y^2 \sec^2\theta = 1$
(b) $x^2 \operatorname{sech}^2 t + y^2 \operatorname{cosech}^2 t = 1$

Exercise 4B

1 $2\sinh 2x$ 2 $\frac{1}{2}\cosh\dfrac{x}{2}$

3 $3\operatorname{sech}^2 3x$ 4 $-2\operatorname{sech} 2x \tanh 2x$

5 $-\frac{1}{3}\operatorname{cosech}\dfrac{x}{3}\coth\dfrac{x}{3}$

6 $e^x(\cosh x + \sinh x)$

7 $6\sinh 3x\cosh 3x$ 8 $3\tanh^2 x \operatorname{sech}^2 x$

9 $-\dfrac{1}{x}\operatorname{cosech}^2(\ln x)$

10 $\coth x$

11 $\sinh 2x + 2x\cosh 2x$

12 $3(x^2\cosh 3x + x^3\sinh 3x)$

13 $\dfrac{1}{\sinh x\cosh x}$ 14 $(\cosh x)e^{\sinh x}$

15 $\dfrac{\cosh x - x\sinh x}{\cosh^2 x}$ 16 $\dfrac{x\sinh x - \cosh x}{x^2}$

17 $3\cosh^2 x \sinh x\, e^{\cosh^3 x}$

18 $-\dfrac{2x\operatorname{cosech}^2 2x + 3\coth 2x}{x^4}$

19 $\dfrac{-\operatorname{cosech}(x^2)(2x^2\coth x^2 + 1)}{x^2}$

20 $\dfrac{\operatorname{sech} x(\operatorname{sech} x + \tanh x)}{\tanh x - \operatorname{sech} x}$ 21 $\dfrac{1}{\sqrt{2}}$

22 $y - \frac{9}{2} = -\frac{2}{7}(x - \ln 2)$

23 $(\ln 3, 0)$, $y = 3(x - \ln 3)$

24 $y = 5$ at $x = -\ln 5$

25 $\dfrac{c}{2}(3e^{-2} - e^2)$

26 max. $(-0.962, 11.18)$,
min. $(0.962, -11.18)$

29 $\dfrac{dy}{dx} = 3\sinh 3x\sin x + \cosh 3x\cos x$

$\dfrac{d^2y}{dx^2} = 8\cosh 3x\sin x + 6\sinh 3x\cos x$

30 $x - \dfrac{x^3}{3} + \dots$

31 $y - 0.905 = 0.181(x - 1.5)$
$y - 0.905 = -5.534(x - 1.5)$

32 $\dfrac{1}{\sqrt{(1+x^2)}}$ 33 $\dfrac{1}{\sqrt{(x^2-4)}}$ 34 $\dfrac{2x}{1-4x^2}$

35 $\dfrac{-1}{x\sqrt{(1-x^2)}}$ 36 $\dfrac{-1}{x\sqrt{(1+x^2)}}$ 37 $\dfrac{2}{1-4x^2}$

38 $\dfrac{-1}{2x\sqrt{(1-x)}}$ 39 $\operatorname{arcosh} x + \dfrac{x}{\sqrt{(x^2-1)}}$

40 $\dfrac{\operatorname{arsinh} x - \frac{x}{\sqrt{(1+x^2)}}}{(\operatorname{arsinh} x)^2}$

41 $\dfrac{2\operatorname{artanh} x}{1-x^2}$

42 $\dfrac{-1}{2x\sqrt{(1-x^2)}\sqrt{(\operatorname{arsech} x)}}$

43 $2xe^{x^2}\operatorname{arsinh} x + \dfrac{e^{x^2}}{\sqrt{(1+x^2)}}$

44 $\dfrac{\frac{1}{x}\operatorname{arcosh} x - \frac{\ln x}{\sqrt{(x^2-1)}}}{(\operatorname{arcosh} x)^2}$

45 $\dfrac{\cos x}{1-\sin^2 x} = \sec x$ 46 $\dfrac{\cosh x}{1-\sinh^2 x}$

47 $\dfrac{\operatorname{arsinh} x\sqrt{(1+x^2)} - \arcsin x\sqrt{(1-x^2)}}{(\operatorname{arsinh} x)^2\sqrt{(1-x^4)}}$

48 $y - x = 0$, $y - \operatorname{arsinh} 1 = \dfrac{1}{\sqrt{2}}(x - 1)$

50 $y - \frac{1}{2}\ln 7 = -\frac{7}{16}(x - \frac{3}{4})$

52 $(0, 3)$ 55 ±0.31 56 $(\ln 2, 4)$

57 $-\frac{1}{2}\operatorname{sech} x\tanh x$

60 (a) $(-1, 0)$ (b) $y + x + 1 = 0$

Differential equations

-P3 book P184 Ex8A Q18-22 Extra Q11, 12, 14, 15, 16

Exercise 8A

Find the general solutions of the differential equations in questions 1–10.

1 $\dfrac{dy}{dx} = \cosh 2x$

2 $\dfrac{dy}{dx} = \operatorname{cosech} \tfrac{1}{3}y$

3 $\tan y \dfrac{dy}{dx} = \cot x$

4 $\dfrac{dy}{dx} = e^{2y} \sec^2 x$

5 $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 + 1}, \quad y > 1$

6 $(1 + x^2)^{\frac{1}{2}} \dfrac{dy}{dx} = y^2 + 4$

7 $e^{-x^2} \dfrac{dy}{dx} = xy$

8 $\dfrac{dy}{dx} = e^{x+y}$

9 $\dfrac{dy}{dx} = \dfrac{y}{x^2 - 1}, \quad x > 1$

10 $x^2 \dfrac{dy}{dx} + \sin^2 y = 0$

Obtain the solution that satisfies the given conditions of the differential equations in questions 11–24.

11 $\dfrac{dy}{dx} = 4y^2, \quad y = \tfrac{1}{2}$ at $x = -2$

12 $\dfrac{dy}{dx} = ye^x, \quad y = 1$ at $x = 0$

13 $\dfrac{dy}{dx} = \tan^2 x, \quad y = 0$ at $x = \tfrac{\pi}{4}$

14 $\dfrac{dy}{dx} = e^{2y+3x}, \quad y = \tfrac{1}{2}$ at $x = \tfrac{1}{3}$

15 $\dfrac{dy}{dx} = \dfrac{y}{x}, \quad x > 0$ and $y = 4$ at $x = 1$

16 $e^x \dfrac{dy}{dx} = y^{\frac{1}{2}}, \quad y = 4$ at $x = 0$

17 $\sin x \dfrac{dy}{dx} = \cosh y, \quad 0 < x < \pi, \quad y = 0$ at $x = \tfrac{\pi}{2}$

18 $\sin x \dfrac{dy}{dx} = \tan y(3\cos x + \sin x), \quad y = \frac{\pi}{6}$ at $x = \frac{\pi}{2}$

19 $(5 - 3\sin x)\dfrac{dy}{dx} = 40\cos x, \quad y = 0$ at $x = \frac{3\pi}{2}$

20 $(1 + \cos^2 x)\dfrac{dy}{dx} = y(y + 1)\sin 2x, \quad y = 2$ at $x = 0$

21 $(1 - x^2)\dfrac{dy}{dx} = xy(1 + y^2), \quad x > 1, \quad y = 1$ at $x = 0$

22 $\dfrac{1}{y}\dfrac{dy}{dx} = x + xy, \quad y = 1$ at $x = 0$

23 $(1 + \cosh 2x)\dfrac{dy}{dx} = \operatorname{sech} y, \quad y = 0$ at $x = 0$

24 $e^{-x^2}\dfrac{dy}{dx} = x(y + 2)^2, \quad y = 0$ at $x = 0$

-P3 book P192 Ex8C Q1-9, 13, 14, 16-18

Exercise 8C

In questions 1–5 the differential equations are exact. Find the general solution of each.

1 $y + x\dfrac{dy}{dx} = x^2$

2 $2xy\dfrac{dy}{dx} + y^2 = x^3$

3 $\dfrac{dy}{dx}\sin x + y\cos x = \tan x$

4 $e^{2x}\dfrac{dy}{dx} + 2e^{2x}y = x\sin x$

5 $\dfrac{y - x\dfrac{dy}{dx}}{y^2} = \cos 2x$

In questions 6–12 find the general solution of each linear differential equation.

6 $\dfrac{dy}{dx} + \dfrac{y}{x} = \cos x$

7 $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = 4x + 3$

8 $\dfrac{dy}{dx} - \dfrac{y}{x} = \ln x$

9 $\dfrac{dy}{dx} + \dfrac{y}{2x} = -x^{\frac{1}{2}}$

10 $\dfrac{dy}{dx} + y\cot x = \cos 3x$

11 $\dfrac{dy}{dx} + 2y\tan x = \sin x$

12 $\dfrac{dy}{dx} - \dfrac{y}{x+1} = x$

13 Given that $x\dfrac{dy}{dx} - 2y = x^3\ln x$, find y in terms of x such that $y = 2$ at $x = 1$.

14 Find y in terms of x given that

$$\frac{dy}{dx} + 2y = \sin x$$

and that the solution curve passes through the origin O.

15 Find the general solution of the differential equation

$$\frac{dy}{dx} - 2y\,\text{cosec}\,x = \tan\frac{x}{2}, \quad 0 < x < \pi$$

16 Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y = e^{-2x}(x^3 + x^{-1}), \quad x > 0$$

If you also know that $y = 0$ at $x = 1$, find y in terms of x.

17 Find y in terms of x given that $x\dfrac{dy}{dx} + 3y = e^x$ and that $y = 1$ at $x = 1$.

18 Solve the differential equation, giving y in terms of x, where $x^2\dfrac{dy}{dx} - xy = 1$ and $y = 2$ at $x = 1$.

-P3 book P196 Ex8D

Exercise 8D

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

2 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$

3 $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$

4 $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 18y = 0$

5 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$

6 $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

7 $3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 4y = 0$

8 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 2y = 0$

9 $6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$

10 $3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 21y = 0$

-

-P3 book P197 Ex8E

Exercise 8E

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

2 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

3 $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

4 $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$

5 $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$

6 $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0$

7 $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0$

8 $9\frac{d^2y}{dx^2} + 30\frac{dy}{dx} + 25y = 0$

9 $\frac{d^2y}{dx^2} - \sqrt{8}\frac{dy}{dx} + 2y = 0$

10 $2\frac{d^2y}{dx^2} + \sqrt{40}\frac{dy}{dx} + 5y = 0$

-P3 book P200 Ex8F

Exercise 8F

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} + y = 0$

2 $\frac{d^2y}{dx^2} + 25y = 0$

3 $4\frac{d^2y}{dx^2} + 9y = 0$

4 $16\frac{d^2y}{dx^2} + 49y = 0$

5 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$

6 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

7 $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0$

8 $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$

9 $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

10 $25\frac{d^2y}{dx^2} - 20\frac{dy}{dx} + 13y = 0$

-P3 book P205 Ex8G Q1-9odds, 17,20,23,26-30

Exercise 8G

Solve each of the differential equations in questions 1–15, giving the general solution.

1 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 12$

2 $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x$

3 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{2x}$

4 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2x - 1$

5 $\frac{d^2y}{dx^2} + y = \cos 2x$

6 $\frac{d^2y}{dx^2} + 9y = e^{\frac{1}{2}x}$

7 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 10x - 12$

8 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \cos x$

9 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 6 - 3x$

10 $\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x}$

11 $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 5$

12 $3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = x$

13 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin 2x$

14 $\frac{d^2y}{dx^2} + 16y = 24$

15 $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = \sin x + \cos x$

In questions 16–25 find the solution subject to the given boundary conditions for each of the following differential equations:

16 $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 12$; $\dfrac{dy}{dx} = 1$ and $y = 0$ at $x = 0$

17 $\dfrac{d^2y}{dx^2} + y = e^x$; $\dfrac{dy}{dx} = y = 0$ at $x = 0$

18 $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = \cos x$; $\dfrac{dy}{dx} = 0$ and $y = 1$ at $x = 0$

19 $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 5y = e^{2x}$; $\dfrac{dy}{dx} = y = 2$ at $x = 0$

20 $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 4x$; $\dfrac{dy}{dx} = y = 0$ at $x = 0$

21 $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 2x + 4$; $y = 1$, $\dfrac{dy}{dx} = 0$ at $x = 0$

22 $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y = 20x - 6$; $y = 0$, $\dfrac{dy}{dx} = 6$ at $x = 0$

23 $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 25y = 6\sin x$; $\dfrac{dy}{dx} = y = 0$ at $x = 0$

24 $\dfrac{d^2y}{dx^2} + 9y = 8\sin x$; $\dfrac{dy}{dx} = y = 0$ at $x = \frac{\pi}{2}$

25 $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 6y = 36x$; $\dfrac{dy}{dx} = 4$ and $y = 0$ at $x = 0$

26 Show that $\frac{1}{2}x \sin x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + y = \cos x.$$

Hence find the general solution.

27 Find the value of the constant k so that kxe^{2x} is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 24y = 4e^{2x}$$

Hence find y in terms of x, given that $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$.

28 Find y in terms of x given that

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 20e^{-x}$$

and that $\frac{dy}{dx} = 3$ and $y = 1$ at $x = 0$.

29 Find the general solution of the differential equation

$$4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + y = 17(\cos x - \sin x)$$

30 For the differential equation $\frac{d^2y}{dx^2} + 4y = 10e^{-x}$ find the solution for which $\frac{dy}{dx} = -1$ and $y = \frac{1}{2}$ at $x = 0$.

Exercise 8A

1 $2y = \sinh x + C$ **2** $3\cosh\frac{1}{3}y = x + C$

3 $\sec y = C\sin x$ **4** $e^{-2y} + 2\tan x = C$

5 $\ln\left|\frac{y-1}{y+1}\right| = 2\arctan x + C$

6 $\arctan\frac{1}{2}y = 2\operatorname{arsinh} x + C$

7 $2\ln|y| - e^{x^2} = C$ **8** $e^x + e^{-y} = C$

9 $Cy^2 = \frac{x-1}{x+1}$ **10** $\frac{1}{x} + \cot y = C$

11 $\frac{1}{y} + 4x + 6 = 0$ **12** $\ln|y| = e^x - 1$

13 $y = \tan x - x + \frac{\pi}{4} - 1$

14 $2e^{3x} = 2e + 3e^{-1} - 3e^{-2y}$

15 $y = 4x,\ x > 0$ **16** $2y^{\frac{1}{2}} = 5 - e^{-x}$

17 $2\arctan e^y = \ln|\tan\frac{x}{2}| + \frac{\pi}{2}$

18 $\ln|\sin y| = 3\ln|\sin x| + x - \ln 2 - \frac{\pi}{2}$

19 $y = -\frac{40}{3}\ln\left|\frac{5 - 3\sin x}{8}\right|$

20 $\frac{y+1}{y} = \frac{3}{4}(1 + \cos^2 x)$

21 $2y^2(1 - x^2) = 1 + y^2$

22 $\frac{x^2}{2} = \ln\left|\frac{2y}{y+1}\right|$ **23** $\tanh x = 2\sinh y$

24 $\frac{1}{y+2} + \frac{1}{2}e^{x^2} = 1$

Exercise 8B

1 $y = 4x + C$ **2** $y^2 = 8x + C$

3 $y = Cx$ **4** $y = \frac{1}{2}e^{2x} + C$

5 $y = \sin x + C$ **6** $y^2 = 2x - x^2 + C$

7 $2y^2 = x^2 + C$

8 $xy = C,\ x > 0,\ y > 0$

Exercise 8C

1 $3xy = x^3 + C$ **2** $4xy^2 = x^4 + C$

3 $y\sin x = \ln|\sec x| + C$

4 $ye^{2x} = \sin x - x\cos x + C$

5 $2\frac{x}{y} = \sin 2x + C$

6 $xy = x\sin x + \cos x + C$

7 $x^2y = x^4 + x^3 + C$

8 $\frac{y}{x} = \frac{1}{2}(\ln|x|)^2 + C$ **9** $yx^{\frac{1}{2}} = C - \frac{1}{2}x^2$

10 $y\sin x = \frac{3}{2}\cos^2 x - \cos^4 x + C$

11 $y\sec^2 x = \sec x + C$

12 $\frac{y}{1+x} = x - \ln|1+x| + C$

13 $y = x^3\ln|x| - x^3 + 3x^2$

14 $y = \frac{1}{5}(2\sin x - \cos x + e^{-2x})$

15 $y\cot^2\frac{x}{2} = 2\ln\left|\sin\frac{x}{2}\right| + C$

16 $ye^{2x} = \frac{x^4}{4} + \ln|x| + C;$

$y = \frac{1}{4}(x^4 - 1)e^{-2x} + e^{-2x}\ln x$

17 $y = [e^x(x^2 - 2x + 2) + 1 - e]x^{-3}$

18 $y = \frac{5x}{2} - \frac{1}{2x}$

Exercise 8D

1 $y = Ae^x + Be^{2x}$ **2** $y = Ae^{-x} + Be^{-3x}$

3 $y = Ae^x + Be^{4x}$ **4** $y = Ae^{2x} + Be^{-9x}$

5 $y = Ae^{4x} + Be^{-2x}$ **6** $y = Ae^{2x} + Be^{-3x}$

7 $y = Ae^{2x} + Be^{-\frac{2}{3}x}$

8 $y = e^x(Ae^{x\sqrt{3}} + Be^{-x\sqrt{3}})$

9 $y = Ae^{\frac{2}{3}x} + Be^{-\frac{3}{2}x}$ **10** $y = Ae^{3x} + Be^{-\frac{7}{3}x}$

Exercise 8E

1 $y = (A + Bx)e^x$ **2** $y = (A + Bx)e^{-2x}$

3 $y = (A + Bx)e^{3x}$ **4** $y = (A + Bx)e^{-4x}$

5 $y = (A + Bx)e^{-\frac{1}{2}x}$ **6** $y = (A + Bx)e^{\frac{1}{3}x}$

7 $y = (A + Bx)e^{\frac{3x}{2}}$ **8** $y = (A + Bx)e^{-\frac{5}{3}x}$

9 $y = (A + Bx)e^{x\sqrt{2}}$

10 $y = (A + Bx)e^{-x\sqrt{\frac{5}{2}}}$

Exercise 8F

1 $y = A\cos x + B\sin x$

2 $y = A\cos 5x + B\sin 5x$

3 $y = A\cos\frac{3x}{2} + B\sin\frac{3x}{2}$

4 $y = A\cos\frac{7x}{4} + B\sin\frac{7x}{4}$

5 $y = e^{x}(A\cos 2x + B\sin 2x)$

6 $y = e^{-2x}(A\cos x + B\sin x)$

7 $y = e^{3x}(A\cos x + B\sin x)$

8 $y = e^{-4x}(A\cos 3x + B\sin 3x)$

9 $y = e^{\frac{1}{2}x}(A\cos x + B\sin x)$

10 $y = e^{\frac{2x}{5}}\left(A\cos\frac{3x}{5} + B\sin\frac{3x}{5}\right)$

Exercise 8G

1 $y = Ae^{x} + Be^{3x} + 4$

2 $y = Ae^{-x} + Be^{-2x} + 2x - 3$

3 $y = (A + Bx)e^{x} + e^{2x}$

4 $y = (A + Bx)e^{-2x} + \frac{1}{2}x - \frac{3}{4}$

5 $y = A\cos x + B\sin x - \frac{1}{3}\cos 2x$

6 $y = A\cos 3x + B\sin 3x + \frac{4}{37}e^{\frac{1}{2}x}$

7 $y = e^{-2x}(A\cos x + B\sin x) + 2x - 4$

8 $y = e^{x}(A\cos x + B\sin x) + \frac{1}{5}\cos x - \frac{2}{5}\sin x$

9 $y = Ae^{x} + Be^{3x} - x + \frac{2}{3}$

10 $y = A + Be^{-x} - xe^{-x}$

11 $y = A + Be^{3x} - \frac{5}{3}x$

12 $y = Ae^{x} + Be^{-\frac{1}{3}x} + 2 - x$

13 $y = e^{-2x}(A\cos x + B\sin x)$
$+ \frac{1}{65}\sin 2x - \frac{8}{65}\cos 2x$

14 $y = A\cos 4x + B\sin 4x + \frac{3}{2}$

15 $y = e^{-\frac{1}{2}x}(A\cos\frac{x}{2} + B\sin\frac{x}{2})$
$+ \frac{1}{10}\sin x - \frac{3}{10}\cos x$

16 $y = -\frac{13}{7}e^{x} + \frac{5}{7}e^{3x} + 4$

17 $y = -\frac{1}{2}(\cos x + \sin x - e^{x})$

18 $y = (1 - \frac{1}{2}x)e^{x} - \frac{1}{2}\sin x$

19 $y = \frac{9}{4}e^{x} + \frac{1}{12}e^{5x} - \frac{1}{3}e^{2x}$

20 $y = 2e^{-x}\cos x + 2x - 2$

21 $y = \frac{1}{2}(e^{-2x} + 1)(x + 1)$

22 $y = e^{-x}(\cos 3x + \frac{5}{3}\sin 3x) + 2x - 1$

23 $y = \frac{1}{68}e^{-3x}(4\cos 4x - \sin 4x)$
$+ \frac{1}{17}(4\sin x - \cos x)$

24 $y = \sin 3x + \sin x$

25 $y = e^{6x} - 8e^{-x} + 6x + 7$

26 $y = A\cos x + B\sin x + \frac{1}{2}x\sin x$

27 $k = -\frac{2}{5}$, $y = \frac{1}{25}(e^{12x} - e^{2x}) - \frac{2}{5}xe^{2x}$

28 $y = e^{-x}(2\sin 2x - 4\cos 2x) + 5e^{-x}$

29 $y = Ae^{\frac{x}{4}} + Be^{x} - 4\cos x - \sin x$

30 $y = \frac{1}{2}(\sin 2x - 3\cos 2x) + 2e^{-x}$

Polar Co-ordinates Questions

Exercise 5A

1 Find polar coordinates of the points with the following Cartesian coordinates.

a $(5,12)$ **b** $(-5, 12)$ **c** $(-5, -12)$

d $(2, -3)$ **e** $(\sqrt{3}, -1)$

2 Convert the following polar coordinates into Cartesian form.

a $\left(6, \frac{\pi}{6}\right)$ **b** $\left(6, -\frac{\pi}{6}\right)$ **c** $\left(6, \frac{3\pi}{4}\right)$

d $\left(10, \frac{5\pi}{4}\right)$ **e** $(2, \pi)$

3 Find Cartesian equations for the following curves, where a is a positive constant.

a $r = 2$ **b** $r = 3\sec\theta$ **c** $r = 5\operatorname{cosec}\theta$

d $r = 4a\tan\theta\sec\theta$ **e** $r = 2a\cos\theta$ **f** $r = 3a\sin\theta$

g $r = 4(1 - \cos 2\theta)$ **h** $r = 2\cos^2\theta$ **i** $r^2 = 1 + \tan^2\theta$

4 Find polar equations for the following curves.

a $x^2 + y^2 = 16$ **b** $xy = 4$ **c** $(x^2 + y^2)^2 = 2xy$

d $x^2 + y^2 - 2x = 0$ **e** $(x + y)^2 = 4$ **f** $x - y = 3$

g $y = 2x$ **h** $y = -\sqrt{3}x + a$ **i** $y = x(x - a)$

Exercise 5B

1 Sketch the following curves.

a $r = 6$ **b** $\theta = \frac{5\pi}{4}$ **c** $\theta = -\frac{\pi}{4}$

d $r = 2\sec\theta$ **e** $r = 3\operatorname{cosec}\theta$ **f** $r = 2\sec\left(\theta - \frac{\pi}{3}\right)$

g $r = a\sin\theta$ **h** $r = a(1 - \cos\theta)$ **i** $r = a\cos 3\theta$

j $r = a(2 + \cos\theta)$ **k** $r = a(6 + \cos\theta)$ **l** $r = a(4 + 3\cos\theta)$

m $r = a(2 + \sin\theta)$ **n** $r = a(6 + \sin\theta)$ **o** $r = a(4 + 3\sin\theta)$

p $r = 2\theta$ **q** $r^2 = a^2\sin\theta$ **r** $r^2 = a^2\sin 2\theta$

2 Sketch the graph with polar equation

$$r = k\sec\left(\frac{\pi}{4} - \theta\right)$$

where k is a positive constant, giving the coordinates of any points of intersection with the coordinate axes in terms of k. **(4 marks)**

3 **a** Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 12 - 5i| = 13$$ **(2 marks)**

b Show that this locus of points can be represented by the polar curve

$$r = 24\cos\theta + 10\sin\theta$$ **(4 marks)**

4 **a** Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z + 4 + 3i| = 5$$ **(2 marks)**

b Show that this locus of points can be represented by the polar curve

$$r = -8\cos\theta - 6\sin\theta$$ **(4 marks)**

Exercise 5C

1 Find the area of the finite region bounded by the curve with the given polar equation and the half-lines $\theta = \alpha$ and $\theta = \beta$.

a $r = a\cos\theta,\ \alpha = 0,\ \beta = \frac{\pi}{2}$ **b** $r = a(1 + \sin\theta),\ \alpha = -\frac{\pi}{2},\ \beta = \frac{\pi}{2}$ **c** $r = a\sin 3\theta,\ \alpha = \frac{\pi}{6},\ \beta = \frac{\pi}{4}$

d $r^2 = a^2\cos 2\theta,\ \alpha = 0,\ \beta = \frac{\pi}{4}$ **e** $r^2 = a^2\tan\theta,\ \alpha = 0,\ \beta = \frac{\pi}{4}$ **f** $r = 2a\theta,\ \alpha = 0,\ \beta = \pi$

g $r = a(3 + 2\cos\theta),\ \alpha = 0,\ \beta = \frac{\pi}{2}$

2 Show that the area enclosed by the curve with polar equation $r = a(p + q\cos\theta)$ is $\frac{2p^2 + q^2}{2}\pi a^2$.

3 Find the area of a single loop of the curve with equation $r = a\cos 3\theta$.

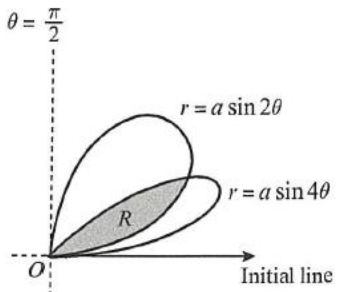
4 A curve has equation $r = a + 5\sin\theta$, $a > 5$. The area enclosed by the curve is $\frac{187\pi}{2}$

Find the value of a. **(5 marks)**

5 The diagram shows the curves with equations $r = a\sin 4\theta$ and $r = a\sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$

The finite region R is contained within both curves.

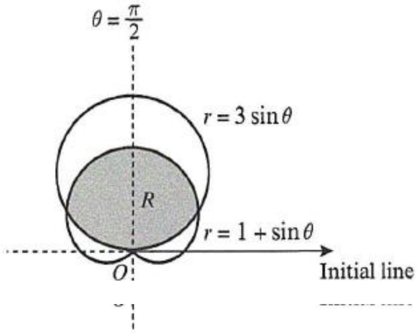
Find the area of R, giving your answer in terms of a. **(8 marks)**



6 The diagram shows the curves with equations $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

The finite region R is contained within both curves.

Find the area of R. **(8 marks)**



7 The set of points, A, is defined by

$$A = \left\{z : -\frac{\pi}{4} \leqslant \arg z \leqslant 0\right\} \cap \{z : |z - 4 + 3\mathrm{i}| \leqslant 5\}$$

a Sketch on an Argand diagram the set of points, A. **(4 marks)**

Given that the locus of points given by the values of z satisfying $|z - 4 + 3\mathrm{i}| = 5$ can be expressed in polar form using the equation $r = 8\cos\theta - 6\sin\theta$,

b find, correct to three significant figures, the area of the region defined by A. **(8 marks)**

8 The set of points, A, is defined by

$$A = \left\{z : \frac{\pi}{2} \leqslant \arg z \leqslant \pi\right\} \cap \{z : |z + 12 - 5\mathrm{i}| \leqslant 13\}$$

a Sketch on an Argand diagram the set of points, A. **(4 marks)**

b Find, correct to three significant figures, the area of the region defined by A. **(8 marks)**

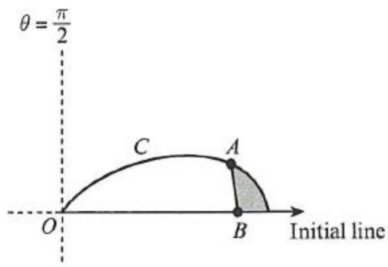
9 The diagram shows the curve C with polar equation

$$r = 1 + \cos 3\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{3}$$

At points A and B, the value of r is $\frac{2+\sqrt{2}}{2}$

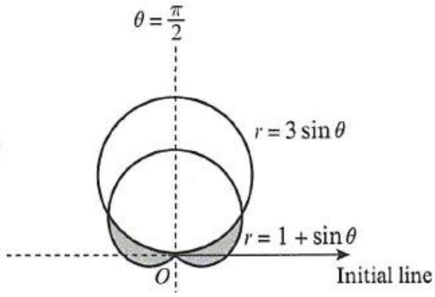
Point A lies on C and point B lies on the initial line.

Find, correct to three significant figures, the finite area bounded by the curve, the line segment AB and the initial line, shown shaded in the diagram. **(9 marks)**



10 The diagram shows the curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

Find the shaded area, giving your answer correct to two decimal places. **(8 marks)**



Exercise 5D

1 Find the points on the cardioid $r = a(1 + \cos\theta)$ where the tangents are perpendicular to the initial line.

2 Find the points on the spiral $r = e^{2\theta}$, $0 \leqslant \theta \leqslant \pi$, where the tangents are

a perpendicular to the initial line **b** parallel to the initial line.

Give your answers to three significant figures.

3 **a** Find the points on the curve $r = a\cos 2\theta$, $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$, where the tangents are parallel to the initial line, giving your answers to three significant figures where appropriate.

b Find the equations of these tangents.

4 Find the points on the curve with equation $r = a(7 + 2\cos\theta)$ where the tangents are parallel to the initial line. **(6 marks)**

5 Find the equations of the tangents to $r = 2 + \cos\theta$ that are perpendicular to the initial line. **(6 marks)**

6 Find the point on the curve with equation $r = a(1 + \tan\theta)$, $0 \leqslant \theta < \frac{\pi}{2}$, where the tangent is perpendicular to the initial line. **(6 marks)**

7 The curve C has polar equation

$$r = 1 + 3\cos\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

The tangent to C at a point A on the curve is parallel to the initial line.

Point O is the pole.

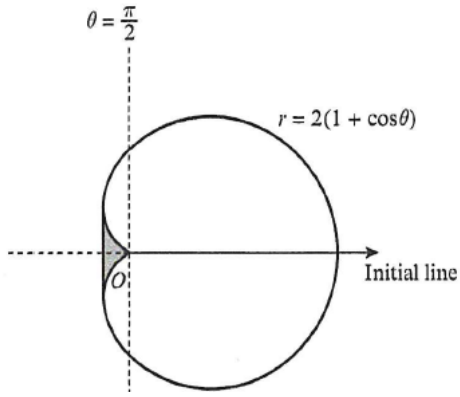
Find the exact length of the line OA. **(7 marks)**

8 The diagram shows a cardioid with polar equation

$r = 2(1 + \cos\theta)$

The shaded area is enclosed by the curve and the vertical line segment which is tangent to the curve and perpendicular to the initial line.

Find the shaded area, correct to three significant figures. **(8 marks)**



Mixed exercise 5

1 Determine the area enclosed by the curve with equation

$r = a(1 + \frac{1}{2}\sin\theta), \quad a > 0, \quad 0 \leqslant \theta < 2\pi,$

giving your answer in terms of a and π. **(6 marks)**

2 **a** Sketch the curve with equation $r = a(1 + \cos\theta)$ for $0 \leqslant \theta \leqslant \pi$, where $a > 0$. **(2 marks)**

b Sketch also the line with equation $r = 2a\sec\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, on the same diagram. **(2 marks)**

c The half-line with equation $\theta = \alpha$, $0 < \alpha < \frac{\pi}{2}$, meets the curve at A and the line with equation $r = 2a\sec\theta$ at B. If O is the pole, find the value of $\cos\alpha$ for which $OB = 2OA$. **(5 marks)**

3 Sketch, in the same diagram, the curves with equations $r = 3\cos\theta$ and $r = 1 + \cos\theta$ and find the area of the region lying inside both curves. **(9 marks)**

4 Find the polar coordinates of the points on $r^2 = a^2\sin 2\theta$ where the tangent is perpendicular to the initial line. **(7 marks)**

5 **a** Shade the region R for which the polar coordinates r, θ satisfy

$r \leqslant 4\cos 2\theta \quad \text{for} \quad -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$ **(2 marks)**

b Find the area of R. **(5 marks)**

6 Sketch the curve with polar equation $r = a(1 - \cos\theta)$, where $a > 0$, stating the polar coordinates of the point on the curve at which r has its maximum value. **(5 marks)**

7 **a** On the same diagram, sketch the curve C_1 with polar equation

$r = 2\cos 2\theta, \quad -\frac{\pi}{4} < \theta \leqslant \frac{\pi}{4}$

and the curve C_2 with polar equation $\theta = \frac{\pi}{12}$ **(3 marks)**

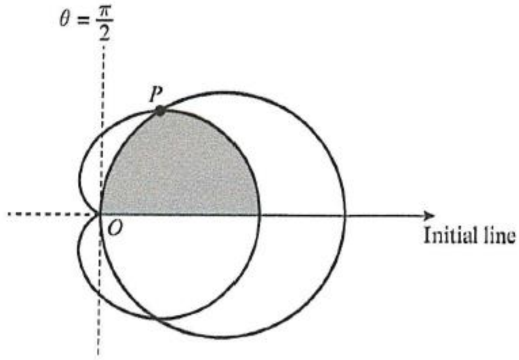
b Find the area of the smaller region bounded by C_1 and C_2. **(6 marks)**

8 **a** Sketch on the same diagram the circle with polar equation $r = 4\cos\theta$ and the line with polar equation $r = 2\sec\theta$. **(4 marks)**

b State polar coordinates for their points of intersection. **(4 marks)**

9 The diagram shows a sketch of the curves with polar equations

$$r = a(1 + \cos\theta) \text{ and } r = 3a\cos\theta,\ a > 0$$



a Find the polar coordinates of the point of intersection P of the two curves. **(4 marks)**

b Find the area, shaded in the figure, bounded by the two curves and by the initial line $\theta = 0$, giving your answer in terms of a and π. **(7 marks)**

10 Obtain a Cartesian equation for the curve with polar equation

a $r^2 = \sec 2\theta$ **(4 marks)**

b $r^2 = \operatorname{cosec} 2\theta$ **(4 marks)**

11 a Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 1 - \mathrm{i}| = \sqrt{2}$$ **(2 marks)**

b Show that this locus of points can be represented by the polar curve

$$r = 2\cos\theta + 2\sin\theta$$ **(4 marks)**

The set of points, A, is defined by

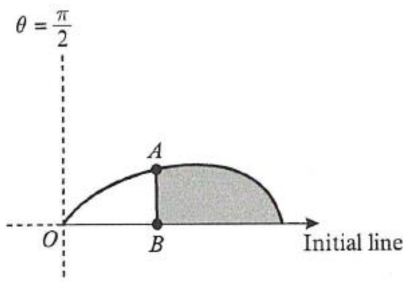
$$A = \left\{z : \frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{2}\right\} \cap \left\{z : |z - 1 - \mathrm{i}| \leqslant \sqrt{2}\right\}$$

c Show, by sketching on your Argand diagram, the set of points, A. **(2 marks)**

d Find, correct to three significant figures, the area of the region defined by A. **(5 marks)**

12 The diagram shows the curve C with polar equation

$$r = 4\cos 2\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{4}$$

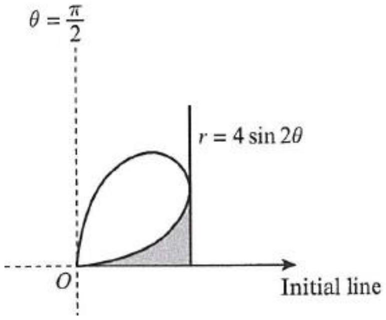


At point A the value of r is 2. Point A lies on C and point B lies on the initial line vertically below A.

Find, correct to three significant figures, the area of the finite region bounded by the curve, the line segment AB and the initial line, shown shaded in the diagram. **(9 marks)**

13 The diagram shows the curve with polar equation

$$r = 4\sin 2\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$



The shaded region is bounded by the curve, the initial line and the tangent to the curve which is perpendicular to the initial line.

Find, correct to two decimal places, the area of the shaded region. **(8 marks)**

SOLUTIONS

Exercise 5A

1 **a** $(13, 1.176)$ **b** $(13, 1.966)$
c $(13, -1.966)$ **d** $(\sqrt{13}, -0.983)$
e $\left(2, -\frac{\pi}{6}\right)$

2 **a** $(3\sqrt{3}, 3)$ **b** $(3\sqrt{3}, -3)$
c $(-3\sqrt{2}, 3\sqrt{2})$ **d** $(-5\sqrt{2}, -5\sqrt{2})$
e $(-2, 0)$

3 **a** $x^2 + y^2 = 4$ **b** $x = 3$
c $y = 5$ **d** $x^2 = 4ay$ or $y = \frac{x^2}{4a}$
e $x^2 + y^2 = 2ax$ or $(x - a)^2 + y^2 = a^2$
f $x^2 + y^2 = 3ay$ or $x^2 + \left(y - \frac{3a}{2}\right)^2 = \frac{9a^2}{4}$
g $(x^2 + y^2)^{\frac{3}{2}} = 8y^2$ **h** $(x^2 + y^2)^{\frac{3}{2}} = 2x^2$
i $x^2 = 1$

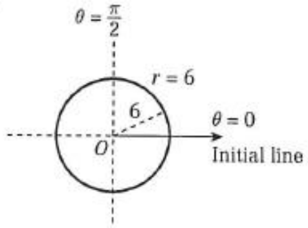
4 **a** $r = 4$ **b** $r^2 = 8\operatorname{cosec} 2\theta$
c $r^2 = \sin 2\theta$ **d** $r = 2\cos\theta$
e $r^2 = \frac{4}{1 + \sin 2\theta}$ **f** $r = \frac{3}{\sqrt{2}}\sec\left(\theta + \frac{\pi}{4}\right)$
g $\theta = \arctan 2$ **h** $r = \frac{a}{2}\operatorname{cosec}\left(\theta + \frac{\pi}{3}\right)$
i $r = \tan\theta\sec\theta + a\sec\theta$

Challenge

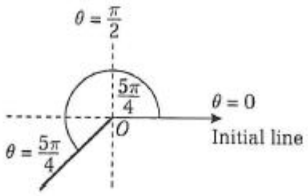
Consider the triangle formed by the two points and the origin and use the cosine rule to find d.

Exercise 5B

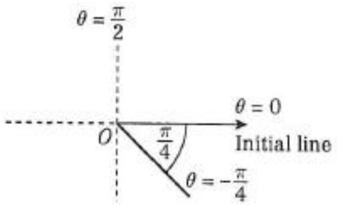
1 **a**



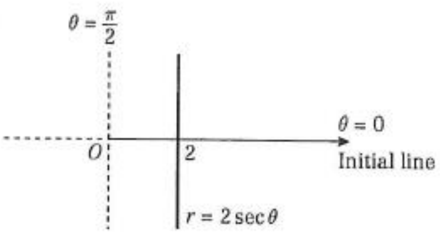
b



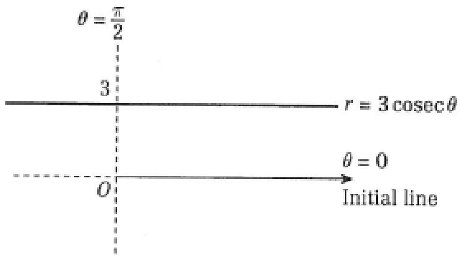
c



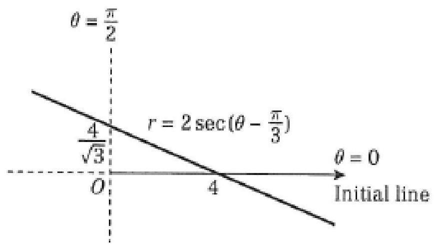
d



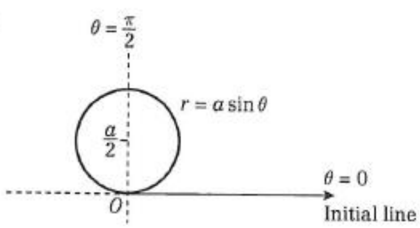
e



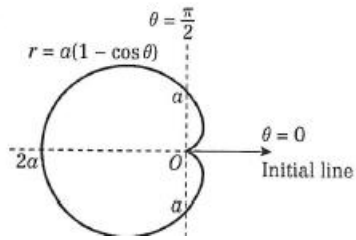
f



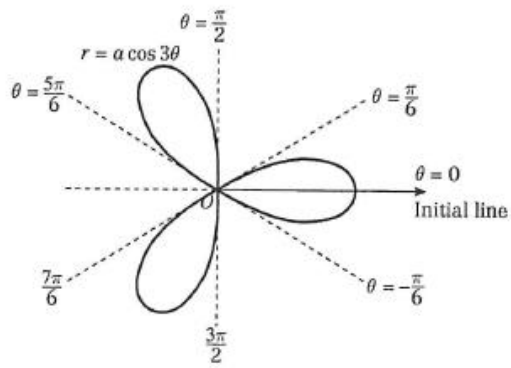
g



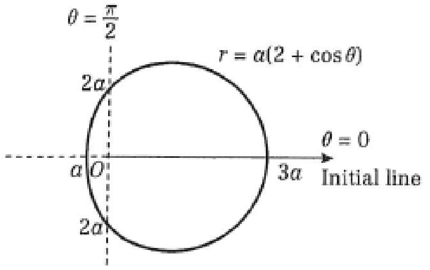
h



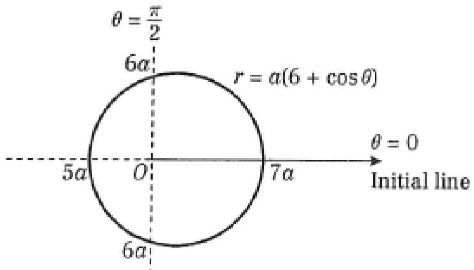
i



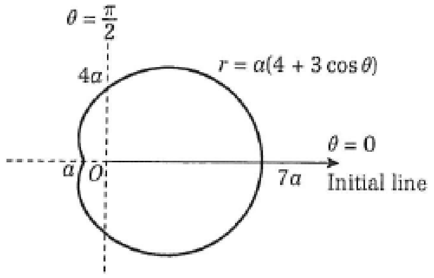
j


$\theta = \frac{\pi}{2}$
$r = a(2 + \cos\theta)$
$2a$
$\theta = 0$
a O
$3a$ Initial line
$2a$

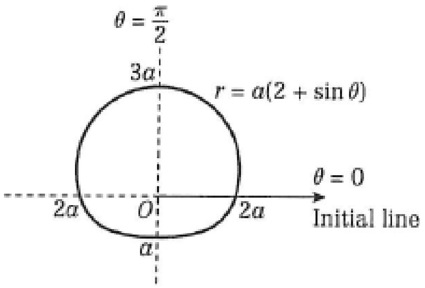
k


$\theta = \frac{\pi}{2}$
$6a$
$r = a(6 + \cos\theta)$
$\theta = 0$
$5a$ O
$7a$ Initial line
$6a$

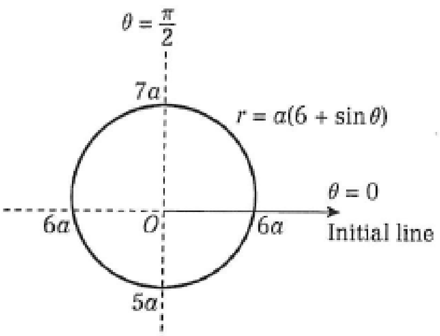
l


$\theta = \frac{\pi}{2}$
$4a$
$r = a(4 + 3\cos\theta)$
$\theta = 0$
a O
$7a$ Initial line

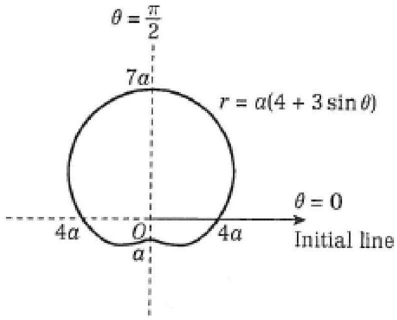
m


$\theta = \frac{\pi}{2}$
$3a$
$r = a(2 + \sin\theta)$
$\theta = 0$
$2a$ O $2a$ Initial line
a

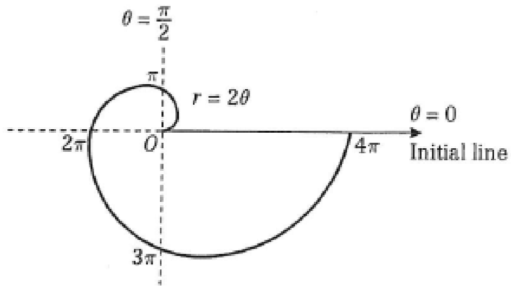
n


$\theta = \frac{\pi}{2}$
$7a$
$r = a(6 + \sin\theta)$
$\theta = 0$
$6a$ O $6a$ Initial line
$5a$

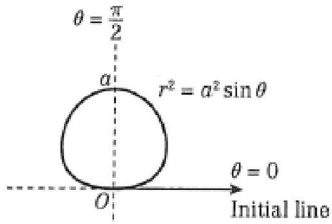
o


$\theta = \frac{\pi}{2}$
$7a$
$r = a(4 + 3\sin\theta)$
$\theta = 0$
$4a$ O $4a$ Initial line
a

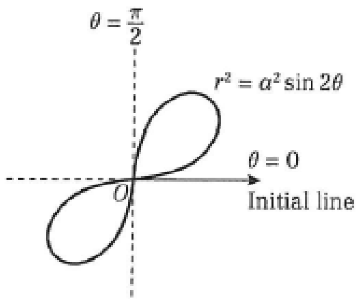
p


$\theta = \frac{\pi}{2}$
π
$r = 2\theta$
$\theta = 0$
2π O
4π Initial line
3π

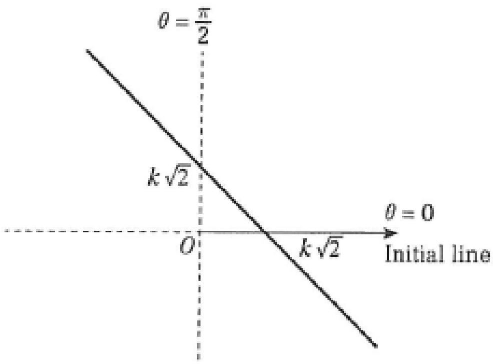
q


$\theta = \frac{\pi}{2}$
a
$r^2 = a^2\sin\theta$
$\theta = 0$
O
Initial line

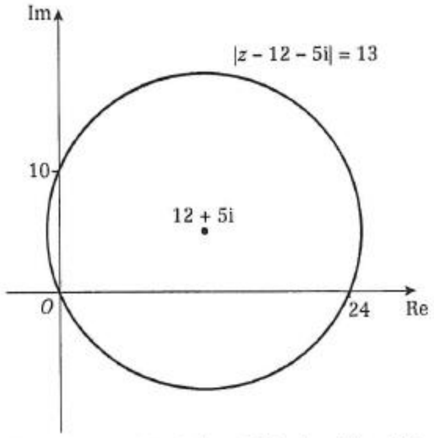
r


$\theta = \frac{\pi}{2}$
$r^2 = a^2\sin 2\theta$
$\theta = 0$
O
Initial line

2

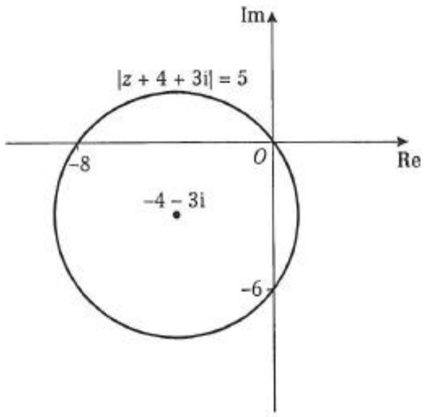

$\theta = \frac{\pi}{2}$
$k\sqrt{2}$
$\theta = 0$
O $k\sqrt{2}$ Initial line

3 a



b Cartesian equation is $(x - 12)^2 + (y - 5)^2 = 169$
Convert to polar coordinates:
$(r\cos\theta - 12)^2 + (r\sin\theta - 5)^2 = 169$
Then rearrange this to get $r = 24\cos\theta + 10\sin\theta$

4 a



b Cartesian equation is $(x + 4)^2 + (y + 3)^2 = 25$
Convert to polar coordinates:
$(r\cos\theta + 4)^2 + (r\sin\theta + 3)^2 = 25$
Then rearrange to get $r = -8\cos\theta - 6\sin\theta$

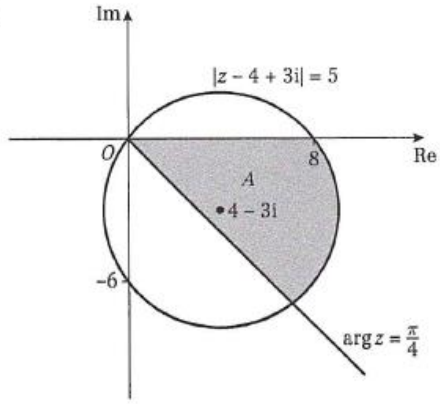
3 $\frac{\pi a^2}{12}$

4 $a = 9$

5 $\frac{a^2}{4}\left(\frac{\pi}{4} - \frac{3\sqrt{3}}{16}\right)$

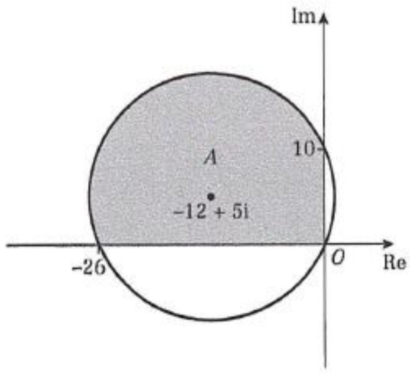
6 $\frac{5\pi}{4}$

7 a



b 35.1

8 a



b 385

9 0.0966

10 0.79

Exercise 5C

1 a $\frac{\pi a^2}{8}$ **b** $\frac{3\pi a^2}{4}$

c $\frac{(\pi + 2)a^2}{48}$ **d** $\frac{a^2}{4}$

e $\frac{a^2\ln\sqrt{2}}{2}$ or $\frac{a^2\ln 2}{4}$ **f** $\frac{2a^2\pi^3}{3}$

g $\frac{a^2}{4}(11\pi + 24)$

2 Area $= 2 \times \frac{1}{2}\int_0^{\pi} a^2(p + q\cos\theta)^2\,d\theta$

$$= a^2\int_0^{\pi}(p^2 + 2pq\cos\theta + q^2\cos^2\theta)\,d\theta$$

$$= a^2\left[p^2\theta + 2pq\sin\theta\right]_0^{\pi} + \frac{a^2q^2}{2}\int_0^{\pi}(\cos 2\theta + 1)\,d\theta$$

$$= a^2p^2\pi + \frac{a^2q^2}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi}$$

$$= a^2p^2\pi + \frac{a^2q^2\pi}{2} = \frac{2p^2 + q^2}{2}\pi a^2$$

Exercise 5D

1 $(2a, 0)$, $\left(\frac{a}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{a}{2}, \frac{-2\pi}{3}\right)$

2 a (9.15, 1.11) **b** (212, 2.68)

3 a $\left(\frac{2a}{3}, \pm 0.421\right)$ **b** $r = \pm\frac{a\sqrt{6}}{9}\operatorname{cosec}\theta$

4 $\left(\frac{15}{2}a, \pm 1.32\right)$

5 $r\cos\theta = 3$ $r\cos\theta = -1$ $r = 3\sec\theta$ $r = -\sec\theta$

6 $\left(2a, \frac{\pi}{4}\right)$

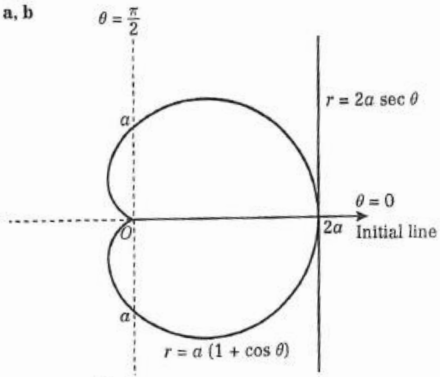
7 $\frac{3 + \sqrt{73}}{4}$

8 0.212

Mixed exercise 5

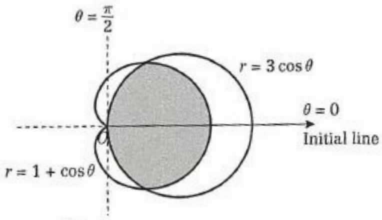
1 $\frac{9\pi a^2}{8}$

2 a, b



c $\cos\alpha = \frac{\sqrt{5}-1}{2}$

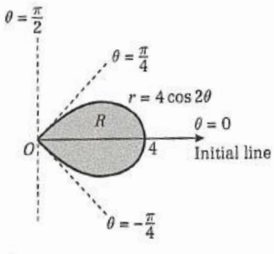
3



Area = $\frac{5\pi}{4}$

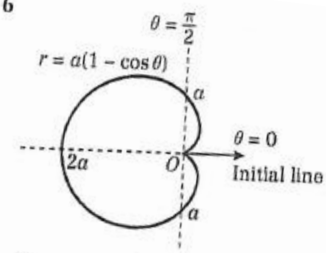
4 $\left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{\pi}{6}\right)$, $\left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{7\pi}{6}\right)$ and $\left(0, \frac{\pi}{2}\right)$

5 a



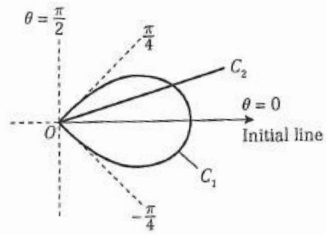
b 2π

6



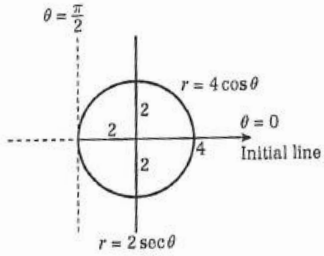
Maximum value at $(2a, \pi)$

7 a



b $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$

8 a

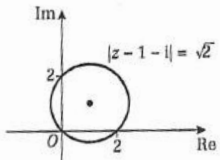


b $\left(2\sqrt{2}, \frac{\pi}{4}\right)$, $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$

9 a $\left(\frac{3}{2}a, \frac{\pi}{3}\right)$ b $\frac{5\pi}{8}a^2$

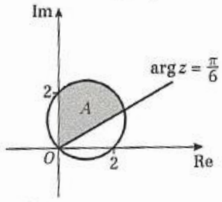
10 a $y^2 = x^2 - 1$ b $y = \frac{1}{2x}$

11 a



b Cartesian equation is $(x-1)^2 - (y+1)^2 = 2$
Convert to polar coordinates:
$(r\cos\theta - 1)^2 + (r\sin\theta - 1)^2 = 2$
Then rearrange to get $r = 2\cos\theta + 2\sin\theta$

c



d 3.59

12 2.09

13 1.52