Section 1: Improper integrals

Section test

1. Which of the following integrals are improper integrals?

   (i) $\displaystyle\int_0^\infty x\,dx$ (ii) $\displaystyle\int_0^1 \frac{1}{x}\,dx$

   (iii) $\displaystyle\int_0^1 \frac{1}{x+1}\,dx$ (iv) $\displaystyle\int_1^3 \frac{1}{x-2}\,dx$

2. The value of the integral $\displaystyle\int_1^\infty \frac{1}{x^{1/3}}\,dx$ is

   (a) $\frac{3}{2}$ (b) 3
   (c) $-\frac{3}{2}$ (d) undefined

3. The value of the integral $\displaystyle\int_0^8 \frac{1}{x^{1/3}}\,dx$ is

   (a) $\frac{3}{2}$ (b) 3
   (c) 6   (d) undefined

4. The value of the integral $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ is

   (a) $\frac{1}{2}$ (b) 1
   (c) $-\frac{1}{2}$ (d) undefined

5. The value of the integral $\displaystyle\int_0^2 \frac{1}{x^3}\,dx$ is

   (a) $\frac{1}{2}$ (b) $-\frac{1}{8}$
   (c) $\frac{1}{8}$ (d) undefined

6. Which of the following integrals can be evaluated?

   (i) $\displaystyle\int_{-2}^\infty \frac{1}{x^2}\,dx$

   (ii) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\,dx$

   (a) (i) only   (b) (ii) only
   (c) both   (d) neither

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Solutions to section test

1. (i) is an improper integral as one of the limits is infinity
    (ii) is an improper integral as the integrand is undefined at $x = 0$
    (iii) is not an improper integral (the integrand is undefined at $x = -1$, but this is not between the limits of the integral
    (iv) is an improper integral as the integrand is undefined at $x = 2$.

2. $\int_1^a \dfrac{1}{x^{1/3}} dx = \int_1^a x^{-1/3} dx$

    $\qquad\qquad = \left[\dfrac{3}{2} x^{2/3} \right]_1^a$

    $\qquad\qquad = \dfrac{3}{2} a^{2/3} - \dfrac{3}{2}$

    As $a \to \infty$, $a^{2/3} \to \infty$, so the integral is undefined.

3. $\int_a^8 \dfrac{1}{x^{1/3}} dx = \int_a^8 x^{-1/3} dx$

    $\qquad\qquad = \left[\dfrac{3}{2} x^{2/3} \right]_a^8$

    $\qquad\qquad = \dfrac{3}{2} \times 8^{2/3} - \dfrac{3}{2} a^{2/3}$

    $\qquad\qquad = 6 - \dfrac{3}{2} a^{2/3}$

    As $a \to 0$, $a^{2/3} \to 0$, so the value of the integral is 6.

4. $\int_1^a \dfrac{1}{x^3} dx = \int_1^a x^{-3} dx$

    $\qquad\qquad = \left[-\dfrac{1}{2} x^{-2} \right]_1^a$

    $\qquad\qquad = -\dfrac{1}{2a^2} + \dfrac{1}{2}$

    As $a \to \infty$, $\dfrac{1}{2a^2} \to 0$, so the value of the integral is $\dfrac{1}{2}$.

5. $\int_a^2 \dfrac{1}{x^3} dx = \int_a^2 x^{-3} dx$

    $\qquad\qquad = \left[-\dfrac{1}{2} x^{-2} \right]_a^2$

    $\qquad\qquad = -\dfrac{1}{8} + \dfrac{1}{2a^2}$

    As $a \to 0$, $\dfrac{1}{2a^2}$ is undefined, so the integral is undefined.

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6. (i) $\int_{-2}^{a} \dfrac{1}{x^2} dx = \int_{-2}^{a} x^{-2} dx + \int_{b}^{c} x^{-2} dx$

$\qquad = \left[-x^{-1} \right]_{-2}^{a} + \left[-x^{-1} \right]_{b}^{c}$

$\qquad = -\dfrac{1}{a} - \dfrac{1}{2} - \dfrac{1}{c} + \dfrac{1}{b}$

As $a \to 0$, $b \to 0$ and $c \to \infty$, $\dfrac{1}{a}$ and $\dfrac{1}{b} \to 0$, and $\dfrac{1}{c}$ is undefined,
so the integral is undefined.

(ii) $\int_{0}^{a} \dfrac{1}{\sqrt{x}} dx = \int_{0}^{a} x^{-1/2} dx$

$\qquad = \left[2x^{1/2} \right]_{0}^{a}$

$\qquad = 2\sqrt{a}$

As $a \to \infty$, $\sqrt{a}$ is undefined, so the integral is undefined.