Maclaurin's Theorem

- P3 book P22 Ex2C Q17, 20-25

Exercise 2C

In questions 1–15, find, in ascending powers of x, the expansions up to and including the term in x^3, simplifying the coefficients. State the set of values of x for which the expansion is valid.

1 $(1+x)^{-2}$ **2** $(1-x)^{-3}$ **3** $(1-x)^{-5}$

4 $(1+x)^{-\frac{1}{2}}$ **5** $(1+x)^{\frac{3}{2}}$ **6** $(1-x)^{\frac{3}{4}}$

7 $(1-3x)^{\frac{1}{3}}$ **8** $(1+3x)^{-\frac{1}{3}}$ **9** $\left(1-\frac{1}{2}x\right)^{-2}$

10 $(1+6x)^{-1}$ **11** $(3+x)^{-1}$ **12** $(2-x)^{-2}$

13 $(4+3x)^{\frac{1}{2}}$ **14** $(8-5x)^{\frac{1}{3}}$ **15** $(100+x)^{-\frac{1}{2}}$

By using partial fractions find, in ascending powers of x, up to and including the term in x^3, expansions for the functions of x given in questions 16–20. State the set of values of x for which the expansion is valid.

16 $\dfrac{2-3x}{1-3x+2x^2}$ **17** $\dfrac{3}{1+x-2x^2}$ **18** $\dfrac{2}{x^2+2x-8}$

19 $\dfrac{1}{x^2+3x+2}$ **20** $\dfrac{8-x}{x^2-x-6}$

21 Given that $|x| < \frac{1}{2}$, expand $(1+x)^2(1-2x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3, simplifying each coefficient.

22 Given that $|x| > 2$ find the first four terms in the series expansion of $\left(1-\dfrac{2}{x}\right)^{\frac{1}{2}}$ in descending powers of x.

By taking $x = 200$ use your series to find a value of $\sqrt{99}$, giving your answer to 7 decimal places. Use your series to find $\sqrt{101}$ to the same degree of accuracy.

23 The series expansion of $(1+px)^q$ in ascending powers of x has coefficients of -10 and 75 in the x and x^2 terms respectively.

(a) Find the value of p and of q.

(b) Find the coefficients of the x^3 and x^4 terms in the expansion.

(c) State the set of values of x for which the series is valid.

24 Given that $|x| < 1$, expand $\left(\frac{1+x}{1-x}\right)^{\frac{1}{3}}$ in ascending powers of x up to and including the term in x^2.

25 The coefficients of x and x^2 in the expansion of $(1+px+qx^2)^{-2}$ in ascending powers of x are 4 and 14 respectively. Find the value of p and of q.

26 The coefficients of the x and x^2 terms in the expansion of $(1+px)^q$ in ascending powers of x are -6 and 6 respectively.

(a) Find the value of p and of q.

(b) Find the x^3 term and the x^4 term in the expansion.

(c) State the set of values of x for which the expansion is valid.

- P3 book P26 Ex2D

Exercise 2D

Using Maclaurin's expansion, and differentiation, show that:

1 $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^r \frac{x^r}{r!} + \cdots$

2 $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$

3 $e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots + \frac{2^r x^r}{r!} + \cdots$

4 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \cdots$

5 $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^r}{r} - \cdots$

Find the first three non-zero terms in the Maclaurin expansion of the function given in ascending powers of x:

6 $\tan x$

7 $\sin^2 x$

8 $\ln\left(\frac{1+x}{1-x}\right)$, $|x| < 1$

9 $(1-2x^2)^{\frac{1}{2}}$

10 $e^x \cos x$

Exercise 2E

1 Given that x is small, find the constants A and B such that

$$(x + \sin x)\cos x \approx Ax + Bx^3$$

2 Given that x is small, find the constants C and D such that

$$\tan x \approx Cx + Dx^3$$

3 Find $\lim_{x \to 0}\left(\frac{\sin(\frac{\pi}{6} + x) - \sin\frac{\pi}{6}}{\sin 2x}\right)$.

4 Given that x is small, show that

$$\frac{\sin x - x\cos x}{x^3} \approx \frac{1}{3}$$

5 Given that x is so small that terms in x^3 and higher powers of x may be disregarded, show that

$$\ln\left[\frac{(1+2x)^2}{1-3x}\right] = 7x + \frac{1}{2}x^2$$

6 Show that for small x:

$$\frac{(1+x)^{\frac{1}{2}}}{(1-x)^2} \approx 1 + \frac{5}{2}x + \frac{31}{8}x^2$$

7 Given that x takes a value near $\frac{\pi}{2}$, explain why $\cos x \approx \frac{\pi}{2} - x$. Use this approximation to find (to 2 decimal places) the smallest positive root of the equation

$$\cos x = \frac{x}{10}$$

8 Given that x is small, show that

$$e^{\sin x} = 1 + x + \tfrac{1}{2}x^2 + Ax^3$$

and determine the value of A. You may assume that terms in x^4 and higher powers of x can be disregarded.

9 Evaluate: (a) $\lim_{x \to 0}\left(\dfrac{\sin x - x}{\sin x - x\cos x}\right)$ (b) $\lim_{x \to 0}\left(\dfrac{\ln(1+x) - x}{\sin^2 x}\right)$.

10 Given that x is small and that terms in x^4 and higher powers of x may be disregarded, show that

$$\ln(\sec x + \tan x) = x + \tfrac{1}{6}x^3$$

ANSWERS

Exercise 2C

1 $1 - 2x + 3x^2 - 4x^3 + \ldots$, $|x| < 1$

2 $1 + 3x + 6x^2 + 10x^3 + \ldots$, $|x| < 1$

3 $1 + 5x + 15x^2 + 35x^3 + \ldots$, $|x| < 1$

4 $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \ldots$, $|x| < 1$

5 $1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 \ldots$, $|x| < 1$

6 $1 - \frac{3}{4}x - \frac{3}{32}x^2 - \frac{5}{128}x^3 \ldots$ $|x| < 1$

7 $1 - x - x^2 - \frac{5}{3}x^3 - \ldots$ $|x| < \frac{1}{3}$

8 $1 - x + 2x^2 - \frac{14}{3}x^3 + \ldots$ $|x| < \frac{1}{3}$

9 $1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \ldots$ $|x| < 2$

10 $1 - 6x + 36x^2 - 216x^3 + \ldots$ $|x| < \frac{1}{6}$

11 $\frac{1}{3} - \frac{x}{9} + \frac{x^2}{27} - \frac{x^3}{81} + \ldots$ $|x| < 3$

12 $\frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \ldots$ $|x| < 2$

13 $2 + \frac{3}{4}x - \frac{9}{64}x^2 + \frac{27}{512}x^3 - \ldots$ $|x| < \frac{4}{3}$

14 $2 - \frac{5}{12}x - \frac{25}{288}x^2 - \frac{625}{20736}x^3 - \ldots$ $|x| < \frac{8}{5}$

15 $\frac{1}{10} - \frac{1}{2000}x + \frac{3}{800\,000}x^2 - \frac{1}{32\,000\,000}x^3 + \ldots$ $|x| < 100$

16 $2 + 3x + 5x^2 + 9x^3 + \ldots$ $|x| < \frac{1}{2}$

17 $3 - 3x + 9x^2 - 15x^3 + \ldots$ $|x| < \frac{1}{2}$

18 $-\frac{1}{4} - \frac{x}{16} - \frac{3}{64}x^2 - \frac{5}{256}x^3 - \ldots$ $|x| < 2$

19 $\frac{1}{2} - \frac{3}{4}x + \frac{7}{8}x^2 - \frac{15}{16}x^3 + \ldots$ $|x| < 1$

20 $-\frac{4}{3} + \frac{7}{18}x - \frac{31}{108}x^2 + \frac{73}{648}x^3 - \ldots$ $|x| < 2$

21 $1 + 3x + \frac{9}{2}x^2 + \frac{13}{2}x^3 + \ldots$

22 $1 - x^{-1} - \frac{1}{2}x^{-2} - \frac{1}{2}x^{-3}$, 9.949 874 4; 10.049 875 6

23 (a) $p = 5$, $q = -2$ (b) -500, 3125 (c) $|x| < \frac{1}{5}$

24 $1 + \frac{2}{3}x + \frac{2}{9}x^2$

25 $p = -2$, $q = -1$

26 (a) $p = -4$, $q = \frac{3}{2}$ (b) $4x^3$, $6x^4$ (c) $|x| < \frac{1}{4}$

Exercise 2D

6 $x + \frac{1}{3}x^3 + \frac{2}{15}x^5 \ldots$

7 $x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 \ldots$

8 $2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 \ldots$

9 $1 - x^2 - \frac{1}{2}x^4 \ldots$

10 $1 + x - \frac{1}{3}x^3 \ldots$

Exercise 2E

1 $A = 2$, $B = -\frac{7}{6}$ **2** $C = 1$, $D = \frac{1}{3}$

3 $\frac{1}{4}\sqrt{3}$ **7** 1.43 **8** $A = 0$

9 (a) $-\frac{1}{2}$ (b) $-\frac{1}{2}$