Induction

- P4 book P279 Ex8A Q1-6, 9, 17, 20, 25, 29, 30, 34 Extra Q8, 13,15

Exercise 8A

In questions 1–15, use the method of mathematical induction to prove the result given.

1 $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

2 $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

3 $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$

4 $\sum_{r=1}^{n} r(r!) = (n+1)! - 1$

5 $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)} = \frac{n}{2(n+2)}$

6 $1 + 2x + 3x^2 + \ldots + nx^{n-1} = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{1-x}$

7 $\sum_{r=1}^{n} r(3r-1) = n^2(n+1)$

8 $\sum_{r=1}^{n} \sin^2(2r-1)\theta = \frac{1}{2}n - \frac{\sin 4n\theta}{4\sin 2\theta}$

9 $\sum_{r=1}^{n} \frac{3^r(r+1)}{(r+4)!} = \frac{1}{8} - \frac{3^{n+1}}{(n+4)!}$

10 $\sum_{r=1}^{n} r(r+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5)$

11 $\sum_{r=1}^{n} \frac{2r-1}{2^{r-1}} = 6 - \frac{2n+3}{2^{n-1}}$

12 $\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$

13 $\sum_{r=1}^{n} \cos(2r-1)\theta = \frac{\sin n\theta \cos n\theta}{\sin\theta}$

14 $\sum_{r=1}^{n} \operatorname{cosec}(2^r\theta) = \cot\theta - \cot(2^n\theta)$

15 $\sum_{r=1}^{n} \tan r\theta \tan(r+1)\theta = \tan(n+1)\theta \cot\theta - n - 1$

16 Given that n is a positive integer, prove that

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$$

17 Given that ${}^nC_r = \frac{n!}{r!\,(n-r)!}$, show that

$${}^{n+1}C_r = {}^nC_r + {}^nC_{r-1}$$

Use the method of mathematical induction to prove that

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$$

where n is a positive integer.

18 Use the method of induction to show that

$$\sum_{r=1}^{2n} r^3 = n^2(2n+1)^2$$

19 Given that n is a positive integer, prove that $n(n+1)(2n+1)$ is divisible by 6.

20 Given that n is a positive integer, prove that $3^{4n+2} + 2^{6n+3}$ is divisible by 17.

21 (a) If n is an odd positive integer, prove that $2^n + 1$ is divisible by 3.

(b) If n is an even positive integer, prove that $2^n - 1$ is divisible by 3.

22 Given that n is an even positive integer, prove that $(2^{2n} - 1)$ is divisible by 5.

23 Given that n is an odd positive integer, prove that $(5^{2n} + 1)$ is divisible by 13.

24 Given that $A_n = 2^{n+2} + 3^{2n+1}$, show that

$$A_{n+1} - 2A_n = 7(3^{2n+1})$$

Hence use the method of mathematical induction to prove that A_n is divisible by 7, where n is any positive integer.

25 Given that m is an odd positive integer, prove that $(m^2 + 3)(m^2 + 15)$ is divisible by 32 for all such values of m.

26 Given that n is a positive integer, prove that $3^{2n} + 11$ is divisible by 4.

27 Given that n is a positive integer, prove that $(3n + 1)7^n - 1$ is divisible by 9.

28 Given that $0 < x < \frac{\pi}{2}$, and n is a positive integer, prove that $(1 - \sin x)^n < 1$.

29 Given that n is a positive integer, use the method of mathematical induction to prove that

$$\sum_{r=1}^{n} r^2 \geqslant n\left(\frac{n+1}{2}\right)^2$$

30 Given that n is a positive integer, prove by induction that

$$1 + 2 + 3 + \ldots + n > \tfrac{1}{2}n^2$$

31 Given that n is a positive integer, prove that

$$\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\ldots+\frac{2n-1}{2n}<n-\frac{1}{2}, \text{ for } n\geqslant 2$$

32 Given that n is a positive integer, prove that

$$\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\ldots+\frac{2n-1}{2n}>\frac{1}{2}n, \text{ for } n\geqslant 2$$

33 Given that n is a positive integer, show by the method of induction that

$$\frac{n}{2}<\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1}<n, \text{ for } n\geqslant 2$$

34 Given that n is a positive integer where $n\geqslant 2$, prove by the method of mathematical induction that

(a) $\displaystyle\sum_{r=1}^{n-1} r^3 < \frac{n^4}{4}$

(b) $\displaystyle\sum_{r=1}^{n} r^3 > \frac{n^4}{4}$

35 Given that n is a positive integer, prove by induction that

$$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$