Complex Numbers Question Booklet A21

-P4 book P92 Ex3A Q1abcegjk,2aegi,3,7 (*A2)

Exercise 3A

1 Write the following in the form (i) $r(\cos\theta + i\sin\theta)$ (ii) $re^{i\theta}$, $-\pi < \theta \leqslant \pi$, giving θ either as a multiple of π or in radians to 3 significant figures.

(a) $5i$ (b) 7 (c) $-3i$ (d) -6

(e) $1 + i\sqrt{3}$ (f) $3\sqrt{3} - 3i$ (g) $-3 + 4i$ (h) $1 - i$

(i) $6 - 8i$ (j) $\dfrac{2}{1 - i\sqrt{3}}$ (k) $\dfrac{8}{\sqrt{3} - i}$ (l) $\dfrac{3 - 2i}{1 + 4i}$

2 Write the following in the form $a + ib$, $a, b \in \mathbb{R}$:

(a) $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (b) $-5\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

(c) $6\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$ (d) $-4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

(e) $2\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right) \times 5\left(\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}\right)$

(f) $\left[3\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)\right]^2$ (g) $\dfrac{7\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}$

(h) $\dfrac{6\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}{2\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)}$ (i) $\left[2\left(\cos\frac{5\pi}{18} + i\sin\frac{5\pi}{18}\right)\right]^3$

(j) $\dfrac{\left[2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^2}{3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}$

3 Simplify, without the use of a calculator

$$\frac{\left(\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}\right)^3}{\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)^4}$$

[L]

7 Use the fact that $e^x \sin 3x = \text{Im}\left(e^x e^{i3x}\right)$ to find

$$\int e^x \sin 3x \, dx$$

Exercise 3A

1 (a) (i) $5\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)$
(ii) $5\mathrm{e}^{\frac{\mathrm{i}\pi}{2}}$

(b) (i) $7(\cos 0+\mathrm{i}\sin 0)$
(ii) $7\mathrm{e}^{0}$

(c) (i) $3\left[\cos\left(-\frac{\pi}{2}\right)+\mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right]$
(ii) $3\mathrm{e}^{-\frac{\mathrm{i}\pi}{2}}$

(d) (i) $6(\cos\pi+\mathrm{i}\sin\pi)$
(ii) $6\mathrm{e}^{\mathrm{i}\pi}$

(e) (i) $2\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$
(ii) $2\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}$

(f) (i) $6\left[\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]$
(ii) $6\mathrm{e}^{-\frac{\mathrm{i}\pi}{6}}$

(g) (i) $5(\cos 2.21+\mathrm{i}\sin 2.21)$
(ii) $5\mathrm{e}^{2.21\mathrm{i}}$

(h) (i) $\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$
(ii) $\sqrt{2}\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}}$

(i) (i) $10[\cos(-0.927)+\mathrm{i}\sin(-0.927)]$
(ii) $10\mathrm{e}^{-0.927\mathrm{i}}$

(j) (i) $\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}$
(ii) $\mathrm{e}^{\frac{\mathrm{i}\pi}{3}}$

(k) (i) $2(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6})$
(ii) $2\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$

(l) (i) $\frac{\sqrt{221}}{17}[\cos(-1.91)+\mathrm{i}\sin(-1.91)]$
(ii) $\frac{\sqrt{221}}{17}\mathrm{e}^{-1.91\mathrm{i}}$

2 (a) $\frac{3}{2}(1+\mathrm{i}\sqrt{3})$ (b) $-\frac{5}{\sqrt{2}}+\frac{5\mathrm{i}}{\sqrt{2}}$
(c) $3\sqrt{3}-3\mathrm{i}$ (d) $4\mathrm{i}$
(e) -10 (f) $-\frac{9}{2}(\sqrt{3}+\mathrm{i})$
(g) $\frac{7}{3\sqrt{2}}(1+\mathrm{i})$ (h) $\frac{3}{\sqrt{2}}(-1+\mathrm{i})$
(i) $4(-\sqrt{3}+\mathrm{i})$ (j) $\frac{2}{3}(\sqrt{3}+\mathrm{i})$

3 $\mathrm{e}^{-\pi\mathrm{i}}=-1$

4 $\sinh z\cosh w+\cosh z\sinh w$

5 $\dfrac{\tanh z+\tanh w}{1+\tanh z\tanh w}$

6 $1-\tanh^2 z$

7 $\frac{1}{10}\mathrm{e}^{x}(\sin 3x-3\cos 3x)+C$

-P4 book P103 Ex3B Q1-6, 9-12,13b, 14-17, 23, 24 * also (*A2) Q18-22

Exercise 3B

1 Use de Moivre's theorem to simplify:

(a) $\left[\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right]^{10}$ (b) $\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right]^{8}$

(c) $\left[\cos\left(-\frac{\pi}{27}\right) + i\sin\left(-\frac{\pi}{27}\right)\right]^{9}$ (d) $\left(\cos\frac{\pi}{18} - i\sin\frac{\pi}{18}\right)^{3}$

2 Express $z = 2(1 - i\sqrt{3})$ in the form $r(\cos\theta + i\sin\theta)$.
Hence find z^8 and $\frac{1}{z^5}$ in the form $a + ib$.

3 Express $z = (1 - i)$ in the form $r(\cos\theta + i\sin\theta)$. Hence find z^4 and $\frac{1}{z^7}$ in the form $a + ib$.

4 Simplify $\dfrac{\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^4}{\left(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}\right)^5}$ [L]

5 Find $\sin 5\theta$ in terms of $\sin\theta$.

6 Find $\sin 3\theta$ in terms of $\sin\theta$.

7 Find $\cos 7\theta$ in terms of $\cos\theta$.

8 Find $\sin 7\theta$ in terms of $\sin\theta$.

9 Find $\tan 3\theta$ in terms of $\tan\theta$.

10 Find $\tan 5\theta$ in terms of $\tan\theta$.

11 Express (a) $\cos^5\theta$ (b) $\cos^6\theta$ (c) $\cos^7\theta$ in terms of cosines of multiples of θ.

12 Express (a) $\sin^4\theta$ in terms of cosines of multiples of θ
(b) (i) $\sin^5\theta$ (ii) $\sin^7\theta$ in terms of sines of multiples of θ.

13 Find (a) $\int \sin^4\theta\,d\theta$ (b) $\int \cos^6\theta\,d\theta$ (see 11b) (c) $\int \sin^4\theta\cos^2\theta\,d\theta$.

14 Find, in the form $re^{i\theta}$, the cube roots of:

(a) i (b) -1 (c) $-5+12i$ (d) $\dfrac{1+i}{1-i}$

15 Solve the equation

$$z^5+1=0$$

16 Find the cube roots of $21+72i$.

17 Find the fourth roots of unity.

18 One root of the equation $2z^3-9z^2+30z-13=0$ is $2+3i$. Find the other two roots.

19 One root of the equation $z^3-10z^2+33z-34=0$ is $4+i$. Find the other two roots.

20 One root of the equation $2z^3-5z^2+12z-5=0$ is $1-2i$. Find the other two roots.

21 One root of the equation $z^4+3z^3+12z-16=0$ is $2i$. Find the other three roots.

22 One root of the equation $2z^4-11z^3+27z^2-25z+7=0$ is $2-i\sqrt{3}$. Find the other three roots.

23 If $2\cos\theta=z+z^{-1}$, prove that, if n is a positive integer,

$$2\cos n\theta=z^n+z^{-n}$$

Hence, or otherwise, solve the equation

$$3z^4-z^3+2z^2-z+3=0$$

given that no root is real. [L]

24 If $z=\cos\theta+i\sin\theta$, show that

$$z^n+z^{-n}=2\cos n\theta \quad\text{and}\quad z^n-z^{-n}=2i\sin n\theta$$

Hence deduce that

$$\cos^6\theta+\sin^6\theta=\tfrac{1}{8}(3\cos 4\theta+5)$$ [L]

ANSWERS

Exercise 3B

1 (a) $\cos 4\pi + \mathrm{i}\sin 4\pi = 1$

(b) $\cos\frac{8\pi}{12} + \mathrm{i}\sin\frac{8\pi}{12} = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$

(c) $\cos\left(-\frac{\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\mathrm{i}$

(d) $\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}$

2 $4\left[\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right]$; $4^8\left(-\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right)$; $\frac{1}{4^5}\left(\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right)$

3 $\sqrt{2}[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)$; -4; $\frac{1}{16}(1 - \mathrm{i})$

4 $\left(\cos\frac{\pi}{9} + \mathrm{i}\sin\frac{\pi}{9}\right)^9 = \cos\pi + \mathrm{i}\sin\pi = -1$

5 $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

6 $3\sin\theta - 4\sin^3\theta$

7 $-7\cos\theta + 56\cos^3\theta - 112\cos^5\theta + 64\cos^7\theta$

8 $7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

9 $\dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

10 $\dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

11 (a) $\frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

(b) $\frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$

(c) $\frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$

12 (a) $\frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$

(b) $\frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

(c) $-\frac{1}{64}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$

13 (a) $\frac{1}{8}\left(\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta\right) + C$

(b) $\frac{1}{32}\left(\frac{1}{6}\sin 6\theta + \frac{3}{2}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right) + C$

(c) $\frac{1}{192}\sin 6\theta + \frac{5}{64}\sin 4\theta - \frac{1}{64}\sin 2\theta + \frac{\theta}{16} + C$

14 (a) $\mathrm{e}^{\mathrm{i}\left(\frac{2k\pi}{3} + \frac{\pi}{6}\right)}$, $k = 0, 1, 2$

(b) $\mathrm{e}^{\mathrm{i}\left(\frac{2k\pi}{3} + \frac{\pi}{3}\right)}$, $k = 0, 1, 2$

(c) $\sqrt[3]{(13)}\mathrm{e}^{\mathrm{i}\left(\frac{2k\pi}{3} + \frac{\alpha}{3}\right)}$ where $\alpha = 1.965$

(d) $\mathrm{e}^{\mathrm{i}\left(\frac{2k\pi}{3} + \frac{\pi}{6}\right)}$, $k = 0, 1, 2$

15 $z = \mathrm{e}^{\left(\frac{2k\pi}{5} + \frac{\pi}{5}\right)\mathrm{i}}$, $k = 0, 1, 2, 3, 4$

16 $\sqrt[3]{75}\,\mathrm{e}^{\mathrm{i}\left(\frac{1.287 + 2k\pi}{3}\right)}$, $k = 0, 1, 2$

17 $1, -1, \mathrm{i}, -\mathrm{i}$

18 $2 - 3\mathrm{i}$, $\frac{1}{2}$

19 $4 - \mathrm{i}$, 2

20 $1 + 2\mathrm{i}$, $\frac{1}{2}$

21 $-2\mathrm{i}$, -4, 1

22 $2 + \mathrm{i}\sqrt{3}$, $\frac{1}{2}$, 1

23 $\dfrac{2 \pm \mathrm{i}\sqrt{5}}{3}$, $-\dfrac{1 \pm \mathrm{i}\sqrt{3}}{2}$