Polar Co-ordinates Questions

Exercise 5A

1 Find polar coordinates of the points with the following Cartesian coordinates.

a (5,12) **b** (−5, 12) **c** (−5, −12)

d (2, −3) **e** $(\sqrt{3}, -1)$

2 Convert the following polar coordinates into Cartesian form.

a $\left(6, \frac{\pi}{6}\right)$ **b** $\left(6, -\frac{\pi}{6}\right)$ **c** $\left(6, \frac{3\pi}{4}\right)$

d $\left(10, \frac{5\pi}{4}\right)$ **e** $(2, \pi)$

3 Find Cartesian equations for the following curves, where a is a positive constant.

a $r = 2$ **b** $r = 3\sec\theta$ **c** $r = 5\operatorname{cosec}\theta$

d $r = 4a\tan\theta\sec\theta$ **e** $r = 2a\cos\theta$ **f** $r = 3a\sin\theta$

g $r = 4(1 - \cos 2\theta)$ **h** $r = 2\cos^2\theta$ **i** $r^2 = 1 + \tan^2\theta$

4 Find polar equations for the following curves.

a $x^2 + y^2 = 16$ **b** $xy = 4$ **c** $(x^2 + y^2)^2 = 2xy$

d $x^2 + y^2 - 2x = 0$ **e** $(x + y)^2 = 4$ **f** $x - y = 3$

g $y = 2x$ **h** $y = -\sqrt{3}x + a$ **i** $y = x(x - a)$

Exercise 5B

1 Sketch the following curves.

a $r = 6$ **b** $\theta = \frac{5\pi}{4}$ **c** $\theta = -\frac{\pi}{4}$

d $r = 2\sec\theta$ **e** $r = 3\operatorname{cosec}\theta$ **f** $r = 2\sec\left(\theta - \frac{\pi}{3}\right)$

g $r = a\sin\theta$ **h** $r = a(1 - \cos\theta)$ **i** $r = a\cos 3\theta$

j $r = a(2 + \cos\theta)$ **k** $r = a(6 + \cos\theta)$ **l** $r = a(4 + 3\cos\theta)$

m $r = a(2 + \sin\theta)$ **n** $r = a(6 + \sin\theta)$ **o** $r = a(4 + 3\sin\theta)$

p $r = 2\theta$ **q** $r^2 = a^2\sin\theta$ **r** $r^2 = a^2\sin 2\theta$

2 Sketch the graph with polar equation

$$r = k\sec\left(\frac{\pi}{4} - \theta\right)$$

where k is a positive constant, giving the coordinates of any points of intersection with the coordinate axes in terms of k. **(4 marks)**

3 **a** Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z - 12 - 5\mathrm{i}| = 13$$ **(2 marks)**

b Show that this locus of points can be represented by the polar curve

$$r = 24\cos\theta + 10\sin\theta$$ **(4 marks)**

4 **a** Show on an Argand diagram the locus of points given by the values of z satisfying

$$|z + 4 + 3\mathrm{i}| = 5$$ **(2 marks)**

b Show that this locus of points can be represented by the polar curve

$$r = -8\cos\theta - 6\sin\theta$$ **(4 marks)**

Exercise 5C

1 Find the area of the finite region bounded by the curve with the given polar equation and the half-lines $\theta = \alpha$ and $\theta = \beta$.

a $r = a\cos\theta,\ \alpha = 0,\ \beta = \frac{\pi}{2}$
b $r = a(1 + \sin\theta),\ \alpha = -\frac{\pi}{2},\ \beta = \frac{\pi}{2}$
c $r = a\sin 3\theta,\ \alpha = \frac{\pi}{6},\ \beta = \frac{\pi}{4}$
d $r^2 = a^2\cos 2\theta,\ \alpha = 0,\ \beta = \frac{\pi}{4}$
e $r^2 = a^2\tan\theta,\ \alpha = 0,\ \beta = \frac{\pi}{4}$
f $r = 2a\theta,\ \alpha = 0,\ \beta = \pi$
g $r = a(3 + 2\cos\theta),\ \alpha = 0,\ \beta = \frac{\pi}{2}$

2 Show that the area enclosed by the curve with polar equation $r = a(p + q\cos\theta)$ is $\frac{2p^2 + q^2}{2}\pi a^2$.

3 Find the area of a single loop of the curve with equation $r = a\cos 3\theta$.

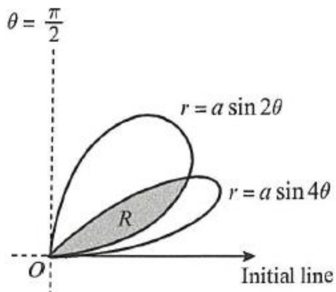
4 A curve has equation $r = a + 5\sin\theta,\ a > 5$. The area enclosed by the curve is $\frac{187\pi}{2}$

Find the value of a. **(5 marks)**

5 The diagram shows the curves with equations $r = a\sin 4\theta$ and $r = a\sin 2\theta$ for $0 \leqslant \theta \leqslant \frac{\pi}{2}$

The finite region R is contained within both curves.

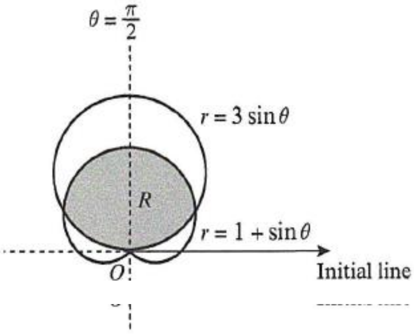
Find the area of R, giving your answer in terms of a. **(8 marks)**



6 The diagram shows the curves with equations $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

The finite region R is contained within both curves.

Find the area of R. **(8 marks)**



7 The set of points, A, is defined by

$$A = \left\{z : -\frac{\pi}{4} \leqslant \arg z \leqslant 0\right\} \cap \{z : |z - 4 + 3i| \leqslant 5\}$$

a Sketch on an Argand diagram the set of points, A. **(4 marks)**

Given that the locus of points given by the values of z satisfying $|z - 4 + 3i| = 5$ can be expressed in polar form using the equation $r = 8\cos\theta - 6\sin\theta$,

b find, correct to three significant figures, the area of the region defined by A. **(8 marks)**

8 The set of points, A, is defined by

$$A = \left\{z : \frac{\pi}{2} \leqslant \arg z \leqslant \pi\right\} \cap \{z : |z + 12 - 5i| \leqslant 13\}$$

a Sketch on an Argand diagram the set of points, A. **(4 marks)**

b Find, correct to three significant figures, the area of the region defined by A. **(8 marks)**

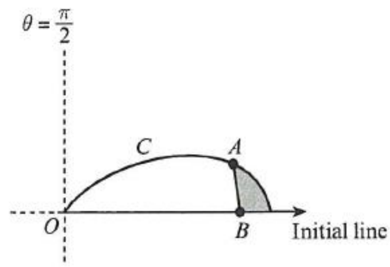
9 The diagram shows the curve C with polar equation

$$r = 1 + \cos 3\theta,\ 0 \leqslant \theta \leqslant \frac{\pi}{3}$$

At points A and B, the value of r is $\frac{2 + \sqrt{2}}{2}$

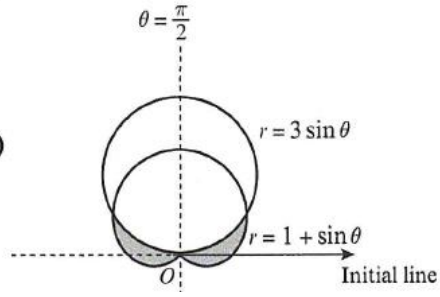
Point A lies on C and point B lies on the initial line.

Find, correct to three significant figures, the finite area bounded by the curve, the line segment AB and the initial line, shown shaded in the diagram. **(9 marks)**



10 The diagram shows the curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

Find the shaded area, giving your answer correct to two decimal places. **(8 marks)**



Exercise 5D

1 Find the points on the cardioid $r = a(1 + \cos\theta)$ where the tangents are perpendicular to the initial line.

2 Find the points on the spiral $r = e^{2\theta}$, $0 \leqslant \theta \leqslant \pi$, where the tangents are

a perpendicular to the initial line **b** parallel to the initial line.

Give your answers to three significant figures.

3 **a** Find the points on the curve $r = a\cos 2\theta$, $-\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$, where the tangents are parallel to the initial line, giving your answers to three significant figures where appropriate.

b Find the equations of these tangents.

4 Find the points on the curve with equation $r = a(7 + 2\cos\theta)$ where the tangents are parallel to the initial line. **(6 marks)**

5 Find the equations of the tangents to $r = 2 + \cos\theta$ that are perpendicular to the initial line. **(6 marks)**

6 Find the point on the curve with equation $r = a(1 + \tan\theta)$, $0 \leqslant \theta < \frac{\pi}{2}$, where the tangent is perpendicular to the initial line. **(6 marks)**

7 The curve C has polar equation

$$r = 1 + 3\cos\theta, \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

The tangent to C at a point A on the curve is parallel to the initial line.

Point O is the pole.

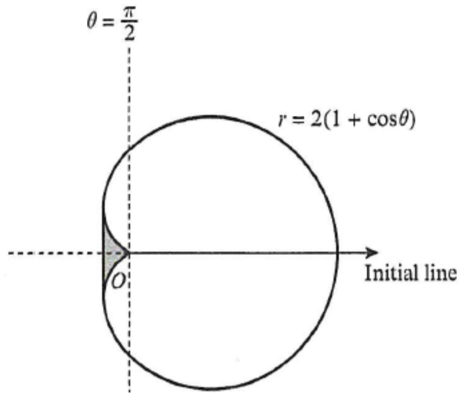
Find the exact length of the line OA. **(7 marks)**

8 The diagram shows a cardioid with polar equation

$r = 2(1 + \cos\theta)$

The shaded area is enclosed by the curve and the vertical line segment which is tangent to the curve and perpendicular to the initial line.

Find the shaded area, correct to three significant figures. **(8 marks)**



Mixed exercise 5

1 Determine the area enclosed by the curve with equation

$r = a(1 + \frac{1}{2}\sin\theta), \quad a > 0, \quad 0 \leqslant \theta < 2\pi,$

giving your answer in terms of a and π. **(6 marks)**

2 **a** Sketch the curve with equation $r = a(1 + \cos\theta)$ for $0 \leqslant \theta \leqslant \pi$, where $a > 0$. **(2 marks)**

b Sketch also the line with equation $r = 2a\sec\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, on the same diagram. **(2 marks)**

c The half-line with equation $\theta = \alpha$, $0 < \alpha < \frac{\pi}{2}$, meets the curve at A and the line with equation $r = 2a\sec\theta$ at B. If O is the pole, find the value of $\cos\alpha$ for which $OB = 2OA$. **(5 marks)**

3 Sketch, in the same diagram, the curves with equations $r = 3\cos\theta$ and $r = 1 + \cos\theta$ and find the area of the region lying inside both curves. **(9 marks)**

4 Find the polar coordinates of the points on $r^2 = a^2\sin 2\theta$ where the tangent is perpendicular to the initial line. **(7 marks)**

5 **a** Shade the region R for which the polar coordinates r, θ satisfy

$r \leqslant 4\cos 2\theta \quad \text{for} \quad -\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{4}$ **(2 marks)**

b Find the area of R. **(5 marks)**

6 Sketch the curve with polar equation $r = a(1 - \cos\theta)$, where $a > 0$, stating the polar coordinates of the point on the curve at which r has its maximum value. **(5 marks)**

7 **a** On the same diagram, sketch the curve C_1 with polar equation

$r = 2\cos 2\theta, \quad -\frac{\pi}{4} < \theta \leqslant \frac{\pi}{4}$

and the curve C_2 with polar equation $\theta = \frac{\pi}{12}$ **(3 marks)**

b Find the area of the smaller region bounded by C_1 and C_2. **(6 marks)**

8 **a** Sketch on the same diagram the circle with polar equation $r = 4\cos\theta$ and the line with polar equation $r = 2\sec\theta$. **(4 marks)**

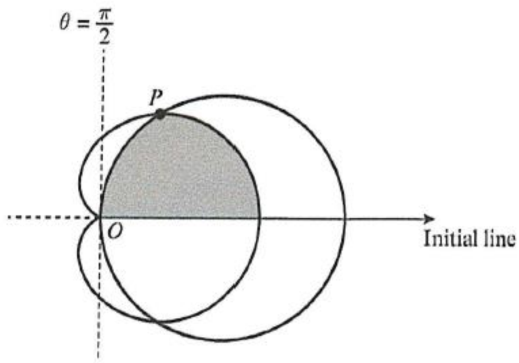
b State polar coordinates for their points of intersection. **(4 marks)**

9 The diagram shows a sketch of the curves with polar equations

$r = a(1 + \cos\theta)$ and $r = 3a\cos\theta$, $a > 0$

a Find the polar coordinates of the point of intersection P of the two curves. **(4 marks)**

b Find the area, shaded in the figure, bounded by the two curves and by the initial line $\theta = 0$, giving your answer in terms of a and π. **(7 marks)**



10 Obtain a Cartesian equation for the curve with polar equation

a $r^2 = \sec 2\theta$ **(4 marks)**

b $r^2 = \operatorname{cosec} 2\theta$ **(4 marks)**

11 **a** Show on an Argand diagram the locus of points given by the values of z satisfying

$|z - 1 - i| = \sqrt{2}$ **(2 marks)**

b Show that this locus of points can be represented by the polar curve

$r = 2\cos\theta + 2\sin\theta$ **(4 marks)**

The set of points, A, is defined by

$$A = \left\{z : \frac{\pi}{6} \leqslant \arg z \leqslant \frac{\pi}{2}\right\} \cap \left\{z : |z - 1 - i| \leqslant \sqrt{2}\right\}$$

c Show, by sketching on your Argand diagram, the set of points, A. **(2 marks)**

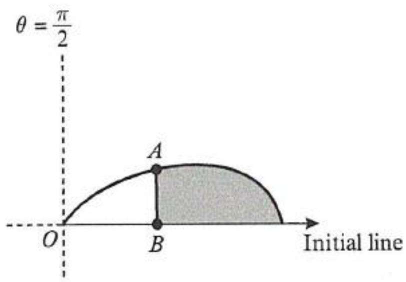
d Find, correct to three significant figures, the area of the region defined by A. **(5 marks)**

12 The diagram shows the curve C with polar equation

$r = 4\cos 2\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{4}$

At point A the value of r is 2. Point A lies on C and point B lies on the initial line vertically below A.

Find, correct to three significant figures, the area of the finite region bounded by the curve, the line segment AB and the initial line, shown shaded in the diagram. **(9 marks)**

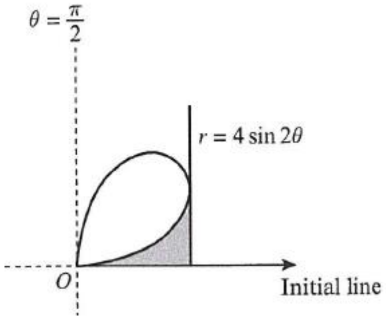


13 The diagram shows the curve with polar equation

$r = 4\sin 2\theta, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$

The shaded region is bounded by the curve, the initial line and the tangent to the curve which is perpendicular to the initial line.

Find, correct to two decimal places, the area of the shaded region. **(8 marks)**



SOLUTIONS

Exercise 5A

1 **a** $(13, 1.176)$ **b** $(13, 1.966)$
c $(13, -1.966)$ **d** $(\sqrt{13}, -0.983)$
e $\left(2, -\frac{\pi}{6}\right)$

2 **a** $(3\sqrt{3}, 3)$ **b** $(3\sqrt{3}, -3)$
c $(-3\sqrt{2}, 3\sqrt{2})$ **d** $(-5\sqrt{2}, -5\sqrt{2})$
e $(-2, 0)$

3 **a** $x^2 + y^2 = 4$ **b** $x = 3$
c $y = 5$ **d** $x^2 = 4ay$ or $y = \frac{x^2}{4a}$
e $x^2 + y^2 = 2ax$ or $(x - a)^2 + y^2 = a^2$
f $x^2 + y^2 = 3ay$ or $x^2 + \left(y - \frac{3a}{2}\right)^2 = \frac{9a^2}{4}$
g $(x^2 + y^2)^{\frac{3}{2}} = 8y^2$ **h** $(x^2 + y^2)^{\frac{3}{2}} = 2x^2$
i $x^2 = 1$

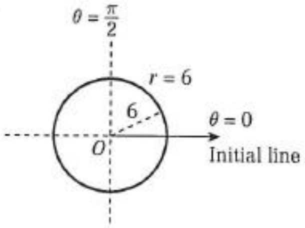
4 **a** $r = 4$ **b** $r^2 = 8 \operatorname{cosec} 2\theta$
c $r^2 = \sin 2\theta$ **d** $r = 2\cos\theta$
e $r^2 = \frac{4}{1 + \sin 2\theta}$ **f** $r = \frac{3}{\sqrt{2}} \sec\left(\theta + \frac{\pi}{4}\right)$
g $\theta = \arctan 2$ **h** $r = \frac{a}{2} \operatorname{cosec}\left(\theta + \frac{\pi}{3}\right)$
i $r = \tan\theta \sec\theta + a\sec\theta$

Challenge

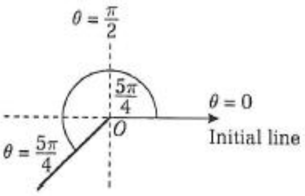
Consider the triangle formed by the two points and the origin and use the cosine rule to find d.

Exercise 5B

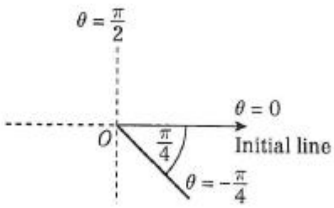
1 **a**



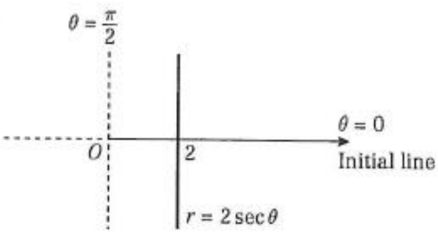
b



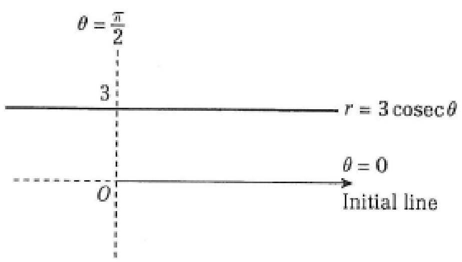
c



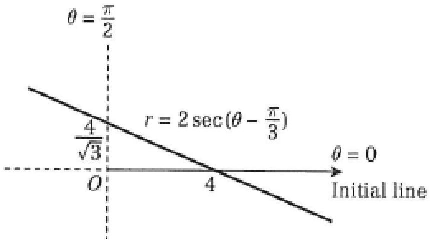
d



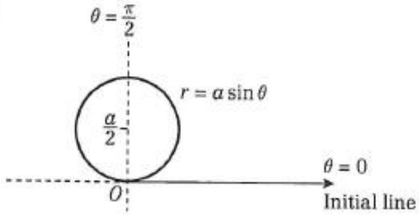
e



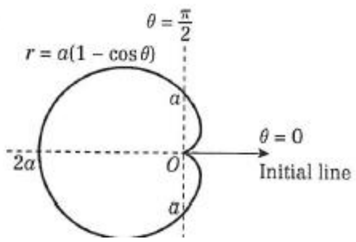
f



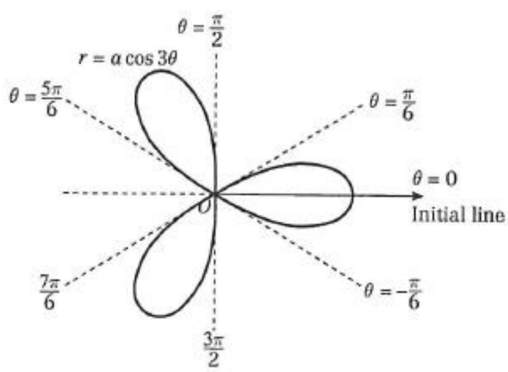
g



h



i



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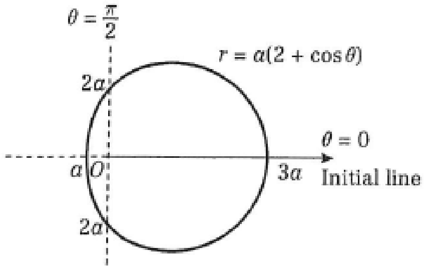
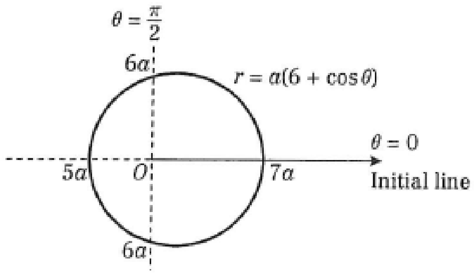
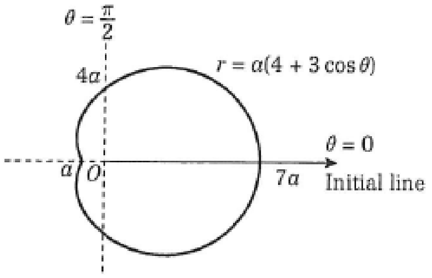

$\theta = \frac{\pi}{2}$
$r = a(2 + \cos\theta)$
$2a$
$\theta = 0$
a O
$3a$
Initial line
$2a$

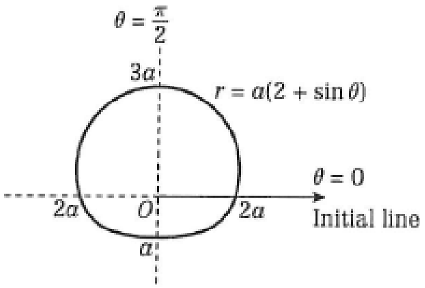
k


$\theta = \frac{\pi}{2}$
$6a$
$r = a(6 + \cos\theta)$
$\theta = 0$
$5a$ O
$7a$
Initial line
$6a$

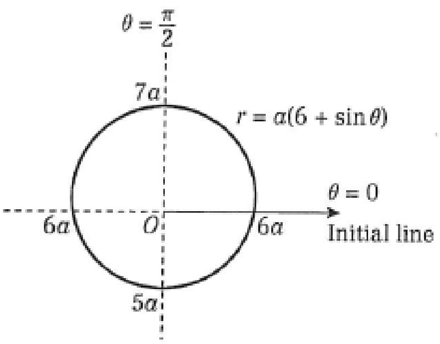
l


$\theta = \frac{\pi}{2}$
$4a$
$r = a(4 + 3\cos\theta)$
$\theta = 0$
a O
$7a$
Initial line

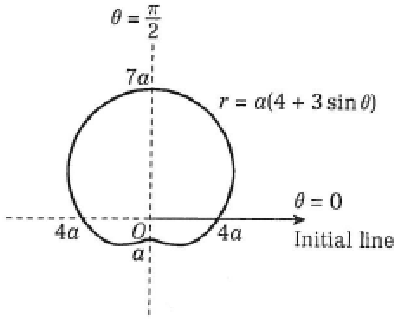
m


$\theta = \frac{\pi}{2}$
$3a$
$r = a(2 + \sin\theta)$
$\theta = 0$
$2a$ O
$2a$
Initial line
a

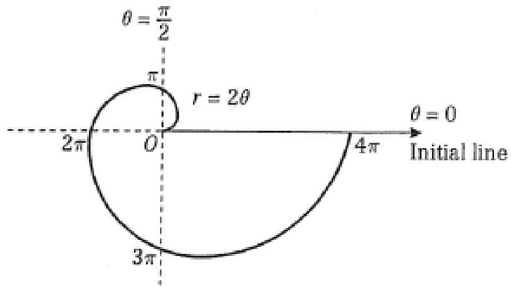
n


$\theta = \frac{\pi}{2}$
$7a$
$r = a(6 + \sin\theta)$
$\theta = 0$
$6a$ O
$6a$
Initial line
$5a$

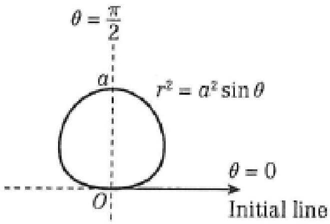
o


$\theta = \frac{\pi}{2}$
$7a$
$r = a(4 + 3\sin\theta)$
$\theta = 0$
$4a$ O
$4a$
Initial line
a

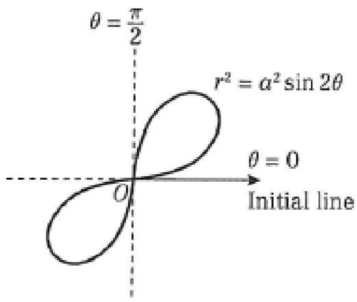
p


$\theta = \frac{\pi}{2}$
π
$r = 2\theta$
$\theta = 0$
2π O
4π
Initial line
3π

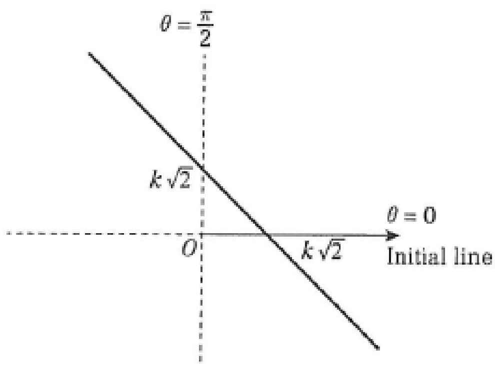
q


$\theta = \frac{\pi}{2}$
a
$r^2 = a^2 \sin\theta$
$\theta = 0$
O
Initial line

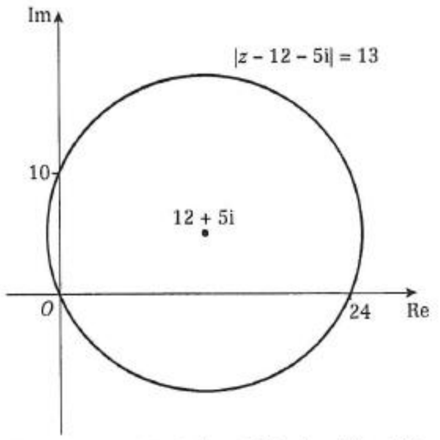
r


$\theta = \frac{\pi}{2}$
$r^2 = a^2 \sin 2\theta$
$\theta = 0$
O
Initial line

2

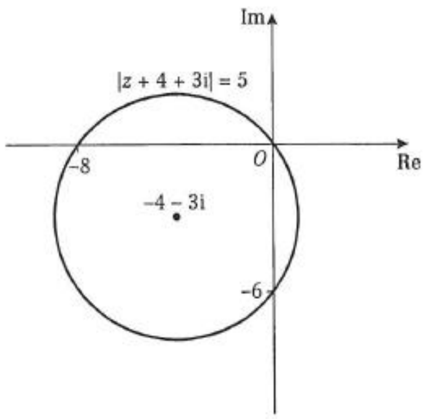

$\theta = \frac{\pi}{2}$
$k\sqrt{2}$
$\theta = 0$
O
$k\sqrt{2}$
Initial line

3 a



b Cartesian equation is $(x - 12)^2 + (y - 5)^2 = 169$
Convert to polar coordinates:
$(r\cos\theta - 12)^2 + (r\sin\theta - 5)^2 = 169$
Then rearrange this to get $r = 24\cos\theta + 10\sin\theta$

4 a



b Cartesian equation is $(x + 4)^2 + (y + 3)^2 = 25$
Convert to polar coordinates:
$(r\cos\theta + 4)^2 + (r\sin\theta + 3)^2 = 25$
Then rearrange to get $r = -8\cos\theta - 6\sin\theta$

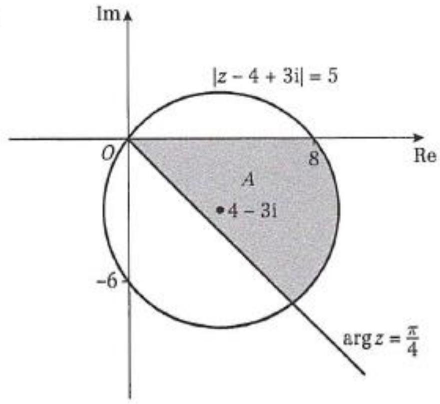
3 $\frac{\pi a^2}{12}$

4 $a = 9$

5 $\frac{a^2}{4}\left(\frac{\pi}{4} - \frac{3\sqrt{3}}{16}\right)$

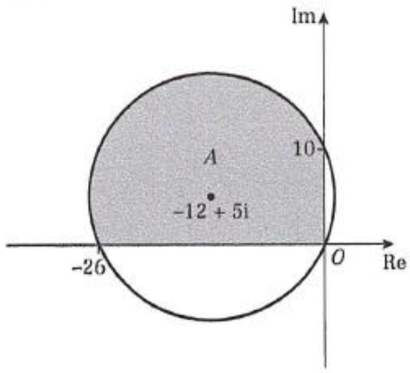
6 $\frac{5\pi}{4}$

7 a



b 35.1

8 a



b 385

9 0.0966

10 0.79

Exercise 5C

1 a $\frac{\pi a^2}{8}$ **b** $\frac{3\pi a^2}{4}$

c $\frac{(\pi + 2)a^2}{48}$ **d** $\frac{a^2}{4}$

e $\frac{a^2 \ln\sqrt{2}}{2}$ or $\frac{a^2 \ln 2}{4}$ **f** $\frac{2a^2\pi^3}{3}$

g $\frac{a^2}{4}(11\pi + 24)$

2 Area $= 2 \times \frac{1}{2}\int_0^{\pi} a^2(p + q\cos\theta)^2\,d\theta$

$$= a^2\int_0^{\pi}(p^2 + 2pq\cos\theta + q^2\cos^2\theta)\,d\theta$$

$$= a^2\left[p^2\theta + 2pq\sin\theta\right]_0^{\pi} + \frac{a^2q^2}{2}\int_0^{\pi}(\cos 2\theta + 1)\,d\theta$$

$$= a^2p^2\pi + \frac{a^2q^2}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\pi}$$

$$= a^2p^2\pi + \frac{a^2q^2\pi}{2} = \frac{2p^2 + q^2}{2}\pi a^2$$

Exercise 5D

1 $(2a, 0)$, $\left(\frac{a}{2}, \frac{2\pi}{3}\right)$ and $\left(\frac{a}{2}, \frac{-2\pi}{3}\right)$

2 a (9.15, 1.11) **b** (212, 2.68)

3 a $\left(\frac{2a}{3}, \pm 0.421\right)$ **b** $r = \pm\frac{a\sqrt{6}}{9}\operatorname{cosec}\theta$

4 $\left(\frac{15}{2}a, \pm 1.32\right)$

5 $r\cos\theta = 3$ $r\cos\theta = -1$ $r = 3\sec\theta$ $r = -\sec\theta$

6 $\left(2a, \frac{\pi}{4}\right)$

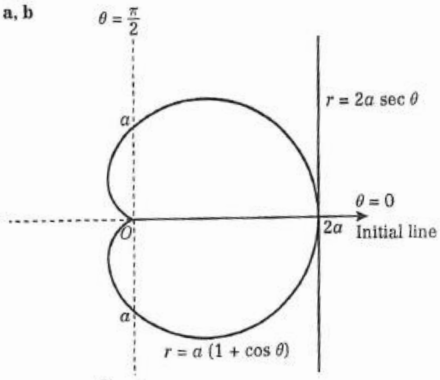
7 $\frac{3 + \sqrt{73}}{4}$

8 0.212

Mixed exercise 5

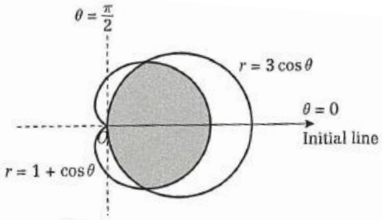
1 $\frac{9\pi a^2}{8}$

2 a, b



c $\cos\alpha = \frac{\sqrt{5} - 1}{2}$

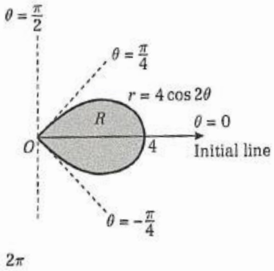
3



Area $= \frac{5\pi}{4}$

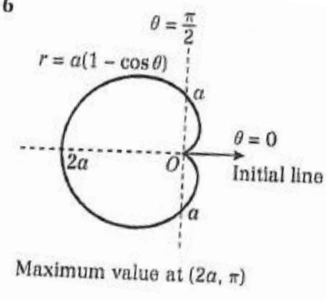
4 $\left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{\pi}{6}\right)$, $\left(a\sqrt{\frac{\sqrt{3}}{2}}, \frac{7\pi}{6}\right)$ and $\left(0, \frac{\pi}{2}\right)$

5 a



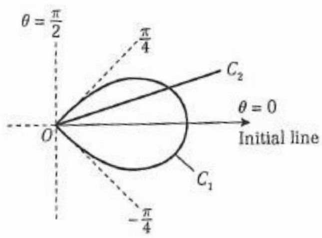
b 2π

6



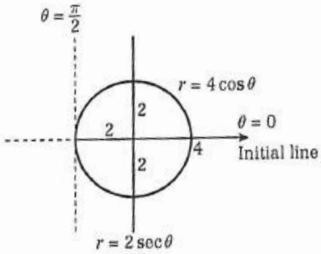
Maximum value at $(2a, \pi)$

7 a



b $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$

8 a

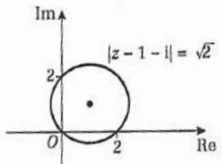


b $\left(2\sqrt{2}, \frac{\pi}{4}\right)$, $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$

9 a $\left(\frac{3}{2}a, \frac{\pi}{3}\right)$ **b** $\frac{5\pi}{8}a^2$

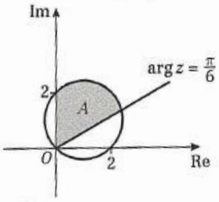
10 a $y^2 = x^2 - 1$ **b** $y = \frac{1}{2x}$

11 a



b Cartesian equation is $(x - 1)^2 - (y + 1)^2 = 2$
Convert to polar coordinates:
$(r\cos\theta - 1)^2 + (r\sin\theta - 1)^2 = 2$
Then rearrange to get $r = 2\cos\theta + 2\sin\theta$

c



d 3.59

12 2.09

13 1.52