Differential equations

-P3 book P184 Ex8A Q18-22 Extra Q11, 12, 14, 15, 16

Exercise 8A

Find the general solutions of the differential equations in questions 1–10.

1 $\dfrac{dy}{dx} = \cosh 2x$

2 $\dfrac{dy}{dx} = \operatorname{cosech} \frac{1}{3}y$

3 $\tan y \dfrac{dy}{dx} = \cot x$

4 $\dfrac{dy}{dx} = e^{2y} \sec^2 x$

5 $\dfrac{dy}{dx} = \dfrac{y^2 - 1}{x^2 + 1}, \quad y > 1$

6 $(1 + x^2)^{\frac{1}{2}} \dfrac{dy}{dx} = y^2 + 4$

7 $e^{-x^2} \dfrac{dy}{dx} = xy$

8 $\dfrac{dy}{dx} = e^{x+y}$

9 $\dfrac{dy}{dx} = \dfrac{y}{x^2 - 1}, \quad x > 1$

10 $x^2 \dfrac{dy}{dx} + \sin^2 y = 0$

Obtain the solution that satisfies the given conditions of the differential equations in questions 11–24.

11 $\dfrac{dy}{dx} = 4y^2, \quad y = \frac{1}{2}$ at $x = -2$

12 $\dfrac{dy}{dx} = ye^x, \quad y = 1$ at $x = 0$

13 $\dfrac{dy}{dx} = \tan^2 x, \quad y = 0$ at $x = \frac{\pi}{4}$

14 $\dfrac{dy}{dx} = e^{2y+3x}, \quad y = \frac{1}{2}$ at $x = \frac{1}{3}$

15 $\dfrac{dy}{dx} = \dfrac{y}{x}, \quad x > 0$ and $y = 4$ at $x = 1$

16 $e^x \dfrac{dy}{dx} = y^{\frac{1}{2}}, \quad y = 4$ at $x = 0$

17 $\sin x \dfrac{dy}{dx} = \cosh y, \quad 0 < x < \pi, \quad y = 0$ at $x = \frac{\pi}{2}$

18 $\sin x \dfrac{dy}{dx} = \tan y(3\cos x + \sin x), \quad y = \frac{\pi}{6}$ at $x = \frac{\pi}{2}$

19 $(5 - 3\sin x)\dfrac{dy}{dx} = 40\cos x, \quad y = 0$ at $x = \frac{3\pi}{2}$

20 $(1 + \cos^2 x)\dfrac{dy}{dx} = y(y+1)\sin 2x, \quad y = 2$ at $x = 0$

21 $(1 - x^2)\dfrac{dy}{dx} = xy(1 + y^2), \quad x > 1, \quad y = 1$ at $x = 0$

22 $\dfrac{1}{y}\dfrac{dy}{dx} = x + xy, \quad y = 1$ at $x = 0$

23 $(1 + \cosh 2x)\dfrac{dy}{dx} = \operatorname{sech} y, \quad y = 0$ at $x = 0$

24 $e^{-x^2}\dfrac{dy}{dx} = x(y+2)^2, \quad y = 0$ at $x = 0$

-P3 book P192 Ex8C Q1-9, 13, 14, 16-18

Exercise 8C

In questions 1–5 the differential equations are exact. Find the general solution of each.

1 $y + x\dfrac{dy}{dx} = x^2$

2 $2xy\dfrac{dy}{dx} + y^2 = x^3$

3 $\dfrac{dy}{dx}\sin x + y\cos x = \tan x$

4 $e^{2x}\dfrac{dy}{dx} + 2e^{2x}y = x\sin x$

5 $\dfrac{y - x\dfrac{dy}{dx}}{y^2} = \cos 2x$

In questions 6–12 find the general solution of each linear differential equation.

6 $\dfrac{dy}{dx} + \dfrac{y}{x} = \cos x$

7 $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = 4x + 3$

8 $\dfrac{dy}{dx} - \dfrac{y}{x} = \ln x$

9 $\dfrac{dy}{dx} + \dfrac{y}{2x} = -x^{\frac{1}{2}}$

10 $\dfrac{dy}{dx} + y\cot x = \cos 3x$

11 $\dfrac{dy}{dx} + 2y\tan x = \sin x$

12 $\dfrac{dy}{dx} - \dfrac{y}{x+1} = x$

13 Given that $x\dfrac{dy}{dx} - 2y = x^3 \ln x$, find y in terms of x such that $y = 2$ at $x = 1$.

14 Find y in terms of x given that

$$\frac{dy}{dx} + 2y = \sin x$$

and that the solution curve passes through the origin O.

15 Find the general solution of the differential equation

$$\frac{dy}{dx} - 2y\operatorname{cosec} x = \tan\frac{x}{2}, \quad 0 < x < \pi$$

16 Find the general solution of the differential equation

$$\frac{dy}{dx} + 2y = e^{-2x}(x^3 + x^{-1}), \quad x > 0$$

If you also know that $y = 0$ at $x = 1$, find y in terms of x.

17 Find y in terms of x given that $x\dfrac{dy}{dx} + 3y = e^x$ and that $y = 1$ at $x = 1$.

18 Solve the differential equation, giving y in terms of x, where $x^2\dfrac{dy}{dx} - xy = 1$ and $y = 2$ at $x = 1$.

-P3 book P196 Ex8D

Exercise 8D

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

2 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 0$

3 $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0$

4 $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 18y = 0$

5 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 8y = 0$

6 $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

7 $3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 4y = 0$

8 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 2y = 0$

9 $6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0$

10 $3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 21y = 0$

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-P3 book P197 Ex8E

Exercise 8E

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$

2 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

3 $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

4 $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$

5 $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$

6 $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0$

7 $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0$

8 $9\frac{d^2y}{dx^2} + 30\frac{dy}{dx} + 25y = 0$

9 $\frac{d^2y}{dx^2} - \sqrt{8}\frac{dy}{dx} + 2y = 0$

10 $2\frac{d^2y}{dx^2} + \sqrt{40}\frac{dy}{dx} + 5y = 0$

-P3 book P200 Ex8F

Exercise 8F

Find the general solution of each of the following differential equations:

1 $\frac{d^2y}{dx^2} + y = 0$

2 $\frac{d^2y}{dx^2} + 25y = 0$

3 $4\frac{d^2y}{dx^2} + 9y = 0$

4 $16\frac{d^2y}{dx^2} + 49y = 0$

5 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$

6 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

7 $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y = 0$

8 $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 25y = 0$

9 $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$

10 $25\frac{d^2y}{dx^2} - 20\frac{dy}{dx} + 13y = 0$

-P3 book P205 Ex8G Q1-9odds, 17,20,23,26-30

Exercise 8G

Solve each of the differential equations in questions 1–15, giving the general solution.

1 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 12$

2 $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x$

3 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{2x}$

4 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2x - 1$

5 $\frac{d^2y}{dx^2} + y = \cos 2x$

6 $\frac{d^2y}{dx^2} + 9y = e^{\frac{1}{2}x}$

7 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 10x - 12$

8 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \cos x$

9 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 6 - 3x$

10 $\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x}$

11 $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 5$

12 $3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = x$

13 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = \sin 2x$

14 $\frac{d^2y}{dx^2} + 16y = 24$

15 $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = \sin x + \cos x$

In questions 16–25 find the solution subject to the given boundary conditions for each of the following differential equations:

16 $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 12$; $\frac{dy}{dx} = 1$ and $y = 0$ at $x = 0$

17 $\frac{d^2y}{dx^2} + y = e^x$; $\frac{dy}{dx} = y = 0$ at $x = 0$

18 $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \cos x$; $\frac{dy}{dx} = 0$ and $y = 1$ at $x = 0$

19 $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 5y = e^{2x}$; $\frac{dy}{dx} = y = 2$ at $x = 0$

20 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 4x$; $\frac{dy}{dx} = y = 0$ at $x = 0$

21 $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2x + 4$; $y = 1$, $\frac{dy}{dx} = 0$ at $x = 0$

22 $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = 20x - 6$; $y = 0$, $\frac{dy}{dx} = 6$ at $x = 0$

23 $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 6\sin x$; $\frac{dy}{dx} = y = 0$ at $x = 0$

24 $\frac{d^2y}{dx^2} + 9y = 8\sin x$; $\frac{dy}{dx} = y = 0$ at $x = \frac{\pi}{2}$

25 $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 36x$; $\frac{dy}{dx} = 4$ and $y = 0$ at $x = 0$

26 Show that $\frac{1}{2}x \sin x$ is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + y = \cos x.$$

Hence find the general solution.

27 Find the value of the constant k so that kxe^{2x} is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 14\frac{dy}{dx} + 24y = 4e^{2x}$$

Hence find y in terms of x, given that $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$.

28 Find y in terms of x given that

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 20e^{-x}$$

and that $\frac{dy}{dx} = 3$ and $y = 1$ at $x = 0$.

29 Find the general solution of the differential equation

$$4\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + y = 17(\cos x - \sin x)$$

30 For the differential equation $\frac{d^2y}{dx^2} + 4y = 10e^{-x}$ find the solution for which $\frac{dy}{dx} = -1$ and $y = \frac{1}{2}$ at $x = 0$.

Exercise 8A

1 $2y = \sinh x + C$ **2** $3\cosh\frac{1}{3}y = x + C$

3 $\sec y = C\sin x$ **4** $e^{-2y} + 2\tan x = C$

5 $\ln\left|\frac{y-1}{y+1}\right| = 2\arctan x + C$

6 $\arctan\frac{1}{2}y = 2\operatorname{arsinh} x + C$

7 $2\ln|y| - e^{x^2} = C$ **8** $e^x + e^{-y} = C$

9 $Cy^2 = \frac{x-1}{x+1}$ **10** $\frac{1}{x} + \cot y = C$

11 $\frac{1}{y} + 4x + 6 = 0$ **12** $\ln|y| = e^x - 1$

13 $y = \tan x - x + \frac{\pi}{4} - 1$

14 $2e^{3x} = 2e + 3e^{-1} - 3e^{-2y}$

15 $y = 4x,\ x > 0$ **16** $2y^{\frac{1}{2}} = 5 - e^{-x}$

17 $2\arctan e^y = \ln|\tan\frac{x}{2}| + \frac{\pi}{2}$

18 $\ln|\sin y| = 3\ln|\sin x| + x - \ln 2 - \frac{\pi}{2}$

19 $y = -\frac{40}{3}\ln\left|\frac{5 - 3\sin x}{8}\right|$

20 $\frac{y+1}{y} = \frac{3}{4}(1 + \cos^2 x)$

21 $2y^2(1 - x^2) = 1 + y^2$

22 $\frac{x^2}{2} = \ln\left|\frac{2y}{y+1}\right|$ **23** $\tanh x = 2\sinh y$

24 $\frac{1}{y+2} + \frac{1}{2}e^{x^2} = 1$

Exercise 8B

1 $y = 4x + C$ **2** $y^2 = 8x + C$

3 $y = Cx$ **4** $y = \frac{1}{2}e^{2x} + C$

5 $y = \sin x + C$ **6** $y^2 = 2x - x^2 + C$

7 $2y^2 = x^2 + C$

8 $xy = C,\ x > 0,\ y > 0$

Exercise 8C

1 $3xy = x^3 + C$ **2** $4xy^2 = x^4 + C$

3 $y\sin x = \ln|\sec x| + C$

4 $ye^{2x} = \sin x - x\cos x + C$

5 $2\frac{x}{y} = \sin 2x + C$

6 $xy = x\sin x + \cos x + C$

7 $x^2y = x^4 + x^3 + C$

8 $\frac{y}{x} = \frac{1}{2}(\ln|x|)^2 + C$ **9** $yx^{\frac{1}{2}} = C - \frac{1}{2}x^2$

10 $y\sin x = \frac{3}{2}\cos^2 x - \cos^4 x + C$

11 $y\sec^2 x = \sec x + C$

12 $\frac{y}{1+x} = x - \ln|1 + x| + C$

13 $y = x^3\ln|x| - x^3 + 3x^2$

14 $y = \frac{1}{5}(2\sin x - \cos x + e^{-2x})$

15 $y\cot^2\frac{x}{2} = 2\ln\left|\sin\frac{x}{2}\right| + C$

16 $ye^{2x} = \frac{x^4}{4} + \ln|x| + C$;

$y = \frac{1}{4}(x^4 - 1)e^{-2x} + e^{-2x}\ln x$

17 $y = [e^x(x^2 - 2x + 2) + 1 - e]x^{-3}$

18 $y = \frac{5x}{2} - \frac{1}{2x}$

Exercise 8D

1 $y = Ae^x + Be^{2x}$ **2** $y = Ae^{-x} + Be^{-3x}$

3 $y = Ae^x + Be^{4x}$ **4** $y = Ae^{2x} + Be^{-9x}$

5 $y = Ae^{4x} + Be^{-2x}$ **6** $y = Ae^{2x} + Be^{-3x}$

7 $y = Ae^{2x} + Be^{-\frac{2}{3}x}$

8 $y = e^x(Ae^{x\sqrt{3}} + Be^{-x\sqrt{3}})$

9 $y = Ae^{\frac{3}{2}x} + Be^{-\frac{3}{2}x}$ **10** $y = Ae^{3x} + Be^{-\frac{7}{3}x}$

Exercise 8E

1 $y = (A + Bx)e^x$ **2** $y = (A + Bx)e^{-2x}$

3 $y = (A + Bx)e^{3x}$ **4** $y = (A + Bx)e^{-4x}$

5 $y = (A + Bx)e^{-\frac{1}{2}x}$ **6** $y = (A + Bx)e^{\frac{1}{3}x}$

7 $y = (A + Bx)e^{\frac{3x}{2}}$ **8** $y = (A + Bx)e^{-\frac{5}{3}x}$

9 $y = (A + Bx)e^{x\sqrt{2}}$

10 $y = (A + Bx)e^{-x\sqrt{\frac{5}{2}}}$

Exercise 8F

1 $y = A\cos x + B\sin x$
2 $y = A\cos 5x + B\sin 5x$
3 $y = A\cos\dfrac{3x}{2} + B\sin\dfrac{3x}{2}$
4 $y = A\cos\dfrac{7x}{4} + B\sin\dfrac{7x}{4}$
5 $y = e^{x}(A\cos 2x + B\sin 2x)$
6 $y = e^{-2x}(A\cos x + B\sin x)$
7 $y = e^{3x}(A\cos x + B\sin x)$
8 $y = e^{-4x}(A\cos 3x + B\sin 3x)$
9 $y = e^{\frac{1}{2}x}(A\cos x + B\sin x)$
10 $y = e^{\frac{2x}{5}}\left(A\cos\dfrac{3x}{5} + B\sin\dfrac{3x}{5}\right)$

Exercise 8G

1 $y = Ae^{x} + Be^{3x} + 4$
2 $y = Ae^{-x} + Be^{-2x} + 2x - 3$
3 $y = (A + Bx)e^{x} + e^{2x}$
4 $y = (A + Bx)e^{-2x} + \frac{1}{2}x - \frac{3}{4}$
5 $y = A\cos x + B\sin x - \frac{1}{3}\cos 2x$
6 $y = A\cos 3x + B\sin 3x + \frac{4}{37}e^{\frac{1}{2}x}$
7 $y = e^{-2x}(A\cos x + B\sin x) + 2x - 4$
8 $y = e^{x}(A\cos x + B\sin x) + \frac{1}{5}\cos x - \frac{2}{5}\sin x$
9 $y = Ae^{x} + Be^{3x} - x + \frac{2}{3}$
10 $y = A + Be^{-x} - xe^{-x}$
11 $y = A + Be^{3x} - \frac{5}{3}x$
12 $y = Ae^{x} + Be^{-\frac{1}{3}x} + 2 - x$
13 $y = e^{-2x}(A\cos x + B\sin x) + \frac{1}{65}\sin 2x - \frac{8}{65}\cos 2x$
14 $y = A\cos 4x + B\sin 4x + \frac{3}{2}$
15 $y = e^{-\frac{1}{2}x}(A\cos\frac{x}{2} + B\sin\frac{x}{2}) + \frac{1}{10}\sin x - \frac{3}{10}\cos x$
16 $y = -\frac{13}{7}e^{x} + \frac{5}{7}e^{3x} + 4$
17 $y = -\frac{1}{2}(\cos x + \sin x - e^{x})$
18 $y = (1 - \frac{1}{2}x)e^{x} - \frac{1}{2}\sin x$
19 $y = \frac{9}{4}e^{x} + \frac{1}{12}e^{5x} - \frac{1}{3}e^{2x}$
20 $y = 2e^{-x}\cos x + 2x - 2$
21 $y = \frac{1}{2}(e^{-2x} + 1)(x + 1)$
22 $y = e^{-x}(\cos 3x + \frac{5}{3}\sin 3x) + 2x - 1$
23 $y = \frac{1}{68}e^{-3x}(4\cos 4x - \sin 4x) + \frac{1}{17}(4\sin x - \cos x)$
24 $y = \sin 3x + \sin x$
25 $y = e^{6x} - 8e^{-x} + 6x + 7$
26 $y = A\cos x + B\sin x + \frac{1}{2}x\sin x$
27 $k = -\frac{2}{5}$, $y = \frac{1}{25}(e^{12x} - e^{2x}) - \frac{2}{5}xe^{2x}$
28 $y = e^{-x}(2\sin 2x - 4\cos 2x) + 5e^{-x}$
29 $y = Ae^{\frac{x}{4}} + Be^{x} - 4\cos x - \sin x$
30 $y = \frac{1}{2}(\sin 2x - 3\cos 2x) + 2e^{-x}$