Hyperbolic functions, Inverse hyp fns, their definitions, derivatives and special integrals

-P3 book P76 Ex4A Q(1-3)alt,4,5,7-17odds,18,20,22,23,25

-P3 book P76 Ex4A Q26,27,29,31-33, 35,38,40

Exercise 4A

1 Express in terms of e:

(a) $\sinh 2$ (b) $\cosh \frac{1}{2}$ (c) $\tanh(-3)$

(d) $\cosh(\sqrt{2})$ (e) $\sinh \pi$ (f) $\tanh 1 - \tanh(-1)$

2 Find, to 3 decimal places, the values of x for which:

(a) $\sinh x = 3$ (b) $\sinh x = -3$ (c) $\cosh x = \frac{3}{2}$

(d) $\cosh x = \sqrt{5}$ (e) $\tanh x = \frac{3}{4}$ (f) $\tanh x = -\frac{2}{3}$

3 Find the value of each of the following, giving each answer to 4 significant figures:

(a) $\cosh 4$ (b) $\sinh \frac{2}{3}$ (c) $\tanh(-2)$

(d) $\sinh(-\frac{1}{2})$ (e) $\cosh \pi$ (f) $\tanh(e^{\frac{1}{2}})$

4 Given that $\cosh x = \frac{5}{3}$, show that $\sinh x = \pm\frac{4}{3}$. Hence find the values of e^x and x.

5 Sketch, in separate diagrams, the curves with equations

(a) $y = \operatorname{cosech} x,\ x \in \mathbb{R},\ x \neq 0$

(b) $y = \coth x,\ x \in \mathbb{R},\ x \neq 0$

Give the equations of the asymptotes to each curve.

6 Sketch, in the same diagram, the curves with equations $y = \sinh 2x$ and $y = \sinh 3x$.

Find the x-coordinates of the points where the curves meet the line $y = 2$, giving your answer to 2 decimal places.

In questions 7–20, prove the given identity and, where appropriate, check the identity independently by using Osborn's rule when you know the comparable trigonometric identity.

7 $\sinh A \equiv -\sinh(-A)$

8 $\sinh 2A \equiv 2 \sinh A \cosh A$

9 $\cosh 2A \equiv 2\cosh^2 A - 1$

10 $\sinh 3A \equiv 3 \sinh A + 4 \sinh^3 A$

11 $\cosh 3A \equiv 4\cosh^3 A - 3\cosh A$

12 $\tanh^2 A + \operatorname{sech}^2 A \equiv 1$

13 $\sinh(A - B) \equiv \sinh A \cosh B - \cosh A \sinh B$

14 $\cosh(A - B) \equiv \cosh A \cosh B - \sinh A \sinh B$

15 $\cosh A + \cosh B \equiv 2 \cosh \dfrac{A+B}{2} \cosh \dfrac{A-B}{2}$

16 $\sinh A + \sinh B \equiv 2 \sinh \dfrac{A+B}{2} \cosh \dfrac{A-B}{2}$

17 $2 \sinh A \sinh B \equiv \cosh(A + B) - \cosh(A - B)$

18 $\dfrac{\cosh x - 1}{\cosh x + 1} \equiv \tanh^2 \dfrac{x}{2}$

19 $\sinh x \equiv \dfrac{2 \tanh \frac{x}{2}}{1 - \tanh^2 \frac{x}{2}}$

20 $\dfrac{\cosh x + \sinh x + 1}{\cosh x + \sinh x - 1} \equiv \coth \dfrac{x}{2}$

21 Given that $\sinh x = \tan \theta$, $0 < \theta < \frac{\pi}{2}$, express $\cosh x$ and $\tanh x$ in terms of θ.

22 Given that $x > 0$, show that

$$\sinh(\ln x) = \frac{x^2 - 1}{2x}$$

Express $\cosh(\ln x)$ in a similar form.

23 Find the value, or values, of x for which

$$4 \sinh x - 3 \cosh x = 5$$

giving your answer, or answers, to 3 significant figures.

24 Given that $\tanh t = \frac{1}{3}$, find the value of e^{2t}. Hence find the exact value of t.

25 Using Maclaurin's expansion for e^x and e^{-x}, express $\sinh x$ and $\cosh x$ as power series in increasing powers of x, up to and including terms in x^5 and x^6 respectively.

26 Given that $\sinh y = x$, show that

$$y = \ln[x + (1 + x^2)^{\frac{1}{2}}]$$

By differentiating this result, show that

$$(1 + x^2)\left(\frac{dy}{dx}\right)^2 = 1$$

27 Solve the equation $2\cosh x + \sinh x = 2$.

28 Solve the equation $13\cosh\theta + 12\sinh\theta = \frac{25}{4}$.

29 Prove that $\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y$.
Given that $a\cosh t + b\sinh t = R\cosh(t + \alpha)$, $a > b > 0$, show that

$$\alpha = \frac{1}{2}\ln\left(\frac{a + b}{a - b}\right)$$

Find R in terms of a and b.

30 Using the definitions of $\sinh x$ and $\cosh x$, in terms of e^x, show that for $|x| < 1$,

$$\operatorname{artanh} x = \frac{1}{2}\ln\frac{1 + x}{1 - x}$$

Hence expand $\operatorname{artanh} x$ in ascending powers of x up to and including the term in x^5.

31 Solve for x the equation

$$3\operatorname{sech}^2 x + 4\tanh x + 1 = 0$$

giving the root as a natural logarithm.

32 Solve the equation

$$\cosh^2 t + \sinh^2 t = 3$$

giving the answers in terms of natural logarithms.

33 Solve the equation

$$4 \tanh t - \operatorname{sech} t = 1$$

giving the answer in terms of a natural logarithm.

34 Prove that $\operatorname{arsinh} x = \ln[x + (1 + x^2)^{\frac{1}{2}}]$.
Given that x is large and positive, show that:

$$\operatorname{arsinh} x \approx \ln 2 + \ln x + \frac{1}{4x^2}$$

35 Solve the equation $\cosh 2x = 3 \sinh x$, giving your answers to 3 significant figures.

36 Given that $p = \frac{1}{2} \ln 2$, find the value of $\tanh p$. Find also the values of $\sinh 2p$, $\cosh 2p$ and $\tanh 2p$.

37 Prove that $\coth A + \operatorname{cosech} A \equiv \coth \dfrac{A}{2}$.

38 Given that $x = \sin\theta \cosh t$ and $y = \cos\theta \sinh t$, find a relation between

(a) x, y and θ (b) x, y and t.

39 Prove that $\lim\limits_{x \to 0} \dfrac{\sinh x - \sin x}{x^3} = \frac{1}{3}$

40 Prove that $\cosh^6 A - \sinh^6 A \equiv 1 + \frac{3}{4} \sinh^2 2A$.
Hence show that

$$8(\cosh^6 A - \sinh^6 A) \equiv 3 \cosh 4A + 5$$

-P3 book P84 Ex4B Q1-19odds,21-25, 27-53odds, 54, 56-59

Exercise 4B

In questions 1–20, differentiate with respect to x:

1 $\cosh 2x$

2 $\sinh \frac{x}{2}$

3 $\tanh 3x$

4 $\operatorname{sech} 2x$

5 $\operatorname{cosech} \frac{x}{3}$

6 $e^x \cosh x$

7 $\sinh^2 3x$

8 $\tanh^3 x$

9 $\coth(\ln x)$

10 $\ln(\sinh x)$

11 $x \sinh 2x$

12 $x^3 \cosh 3x$

13 $\ln(\tanh x)$

14 $e^{\sinh x}$

15 $\frac{x}{\cosh x}$

16 $\frac{\cosh x}{x}$

17 $e^{\cosh^3 x}$

18 $\frac{\coth 2x}{x^3}$

19 $\frac{\operatorname{cosech}(x^2)}{x}$

20 $\ln(\tanh x - \operatorname{sech} x)$

21 Given that $y = \operatorname{arsinh}(x - 1)$, find the value of $\frac{dy}{dx}$ at $x = 2$.

22 Find the equation of the normal at the point where $x = \ln 2$ on the curve $y = \sinh x + 3 \cosh x$.

23 The curve $y = 5 \sinh x - 4 \cosh x$ crosses the x-axis at the point A. Determine the coordinates of A and the equation of the tangent to the curve at A.

24 Find the minimum value of y, where $y = 13 \cosh x + 12 \sinh x$ and the value of x where this occurs.

25 The tangent at the point P with x-coordinate $2c$ on the curve with equation $y = c \cosh \frac{x}{c}$, meets the y-axis at the point Q. Find the distance OQ in terms of c, where O is the origin.

26 Find to 2 decimal places the coordinates of the stationary points on the curve $y = 8 \sinh x - 27 \tanh x$ and determine the nature of these stationary points.

27 Given that $y = A \cosh 3x + B \sinh 3x$, where A and B are constants, show that $\frac{d^2y}{dx^2} - 9y = 0$.

28 Use successive differentiation and Maclaurin's expansion to show that:

- $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$
- $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

29 Given that $y = \cosh 3x \sin x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

30 Find the first two non-zero terms in the series expansion of $\tanh x$ in ascending powers of x.

31 Find an equation of the tangent and an equation of the normal at the point where $x = \frac{3}{2}$ on the curve with equation $y = \tanh x$.

In questions 32–47, differentiate with respect to x:

32 $\operatorname{arsinh} x$

33 $\operatorname{arcosh} \frac{x}{2}$

34 $\operatorname{artanh} x^2$

35 $\operatorname{arsech} x$

36 $\operatorname{arcosech} x$

37 $\operatorname{arcoth} 2x$

38 $\operatorname{arsech} x^{\frac{1}{2}}$

39 $x \operatorname{arcosh} x$

40 $\dfrac{x}{\operatorname{arsinh} x}$

41 $(\operatorname{artanh} x)^2$

42 $(\operatorname{arsech} x)^{\frac{1}{2}}$

43 $e^{x^2} \operatorname{arsinh} x$

44 $\dfrac{\ln x}{\operatorname{arcosh} x}$

45 $\operatorname{artanh}(\sin x)$

46 $\operatorname{artanh}(\sinh x)$

47 $\dfrac{\arcsin x}{\operatorname{arsinh} x}$

48 Find an equation of the tangent to the curve $y = \operatorname{arsinh} x$ at

(i) the origin and

(ii) the point where $x = 1$.

49 Given that $y = (\operatorname{arsinh} x)^2$, show that:

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$$

50 Find an equation of the normal at the point where $x = \frac{3}{4}$ on the curve with equation $y = \operatorname{artanh} x$.

51 Given that $y = \operatorname{arsinh} x$, show that

(a) $y = \ln[x + \sqrt{(1 + x^2)}]$

(b) $(1 + x^2)\left(\dfrac{dy}{dx}\right)^2 = 1$

(c) $(1 + x^2)\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

52 Show that the curve with equation $y = 3\cosh x - x\sinh x$ has a minimum point A on the y-axis. Find the coordinates of A. Show further that the curve has another stationary value between $x = 1.9$ and $x = 2$. Sketch the curve.

53 Show that $y = e^{\operatorname{arsinh} x}$ satisfies the relation

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

54 Given that $y = \sinh x + k\cosh x$, show that the least value of y is $\sqrt{(k^2 - 1)}$ and that this occurs at $x = \frac{1}{2}\ln\left(\frac{k-1}{k+1}\right)$ where k is a constant and $|k| > 1$.

55 Show that $(\cosh x + \sinh x)^k + (\cosh x - \sinh x)^k \equiv 2\cosh kx$, where k is real.
Hence solve the equation

$$(\cosh x + \sinh x)^5 + (\cosh x - \sinh x)^5 = 5$$

giving your answers to 2 decimal places.

56 Find the coordinates of the minimum point on the curve $y = 5\cosh x - 3\sinh x$.

57 Given that $y = \arctan(e^x)$, show that $\frac{dy}{dx} = \frac{1}{2}\,\text{sech}\,x$, and find $\frac{d^2y}{dx^2}$.

58 Given that $\text{artanh}\,x + \text{artanh}\,y = \frac{1}{2}\ln 5$, show that $y = \frac{2-3x}{3-2x}$.

59 Given that $y = \ln\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$, show that $\sinh y = \tan x$ and $\cosh y = \sec x$.

60 For the curve $y = \text{arsinh}\,(x + 1)$, find
(a) the coordinates of its point of inflexion P
(b) the equation of the normal to the curve at P.

ANSWERS

Exercise 4A

1 (a) $\frac{1}{2}(e^2 - e^{-2})$ (b) $\frac{1}{2}(e^{\frac{1}{2}} + e^{-\frac{1}{2}})$
(c) $-\frac{e^3 - e^{-3}}{e^3 + e^{-3}}$ (d) $\frac{1}{2}(e^{\sqrt{2}} + e^{-\sqrt{2}})$
(e) $\frac{1}{2}(e^{\pi} - e^{-\pi})$ (f) $2\left(\frac{e - e^{-1}}{e + e^{-1}}\right)$

2 (a) 1.818 (b) −1.818 (c) ±0.962
(d) ±1.444 (e) 0.973 (f) −0.805

3 (a) 27.31 (b) 0.7172 (c) −0.9640
(d) −0.5211 (e) 11.59 (f) 0.9287

4 $e^x = 3$ or $\frac{1}{3}$, $x = \pm\ln 3$

5 (a) $x = 0, y = 0$ (b) $y = \pm 1, x = 0$

6 0.72, 0.48

21 $\sec\theta, \sin\theta$ 22 $\frac{x^2+1}{2x}$

23 2.37

24 2, $\frac{1}{2}\ln 2$

25 $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

27 $0, -\ln 3$ 28 $-\ln\frac{5}{2}, -\ln 10$

29 $\sqrt{(a^2 - b^2)}$ 30 $x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots$

31 $-\frac{1}{2}\ln 5$ 32 $\pm\frac{1}{2}\ln(3 + 2\sqrt{2})$

33 $\ln\frac{5}{3}$ 35 0.481, 0.881

36 $\frac{1}{3}; \frac{3}{4}, \frac{5}{4}, \frac{3}{5}$

38 (a) $x^2\operatorname{cosec}^2\theta - y^2\sec^2\theta = 1$
(b) $x^2\operatorname{sech}^2 t + y^2\operatorname{cosech}^2 t = 1$

Exercise 4B

1 $2\sinh 2x$ 2 $\frac{1}{2}\cosh\frac{x}{2}$

3 $3\operatorname{sech}^2 3x$ 4 $-2\operatorname{sech} 2x\tanh 2x$

5 $-\frac{1}{3}\operatorname{cosech}\frac{x}{3}\coth\frac{x}{3}$

6 $e^x(\cosh x + \sinh x)$

7 $6\sinh 3x\cosh 3x$ 8 $3\tanh^2 x\operatorname{sech}^2 x$

9 $-\frac{1}{x}\operatorname{cosech}^2(\ln x)$

10 $\coth x$

11 $\sinh 2x + 2x\cosh 2x$

12 $3(x^2\cosh 3x + x^3\sinh 3x)$

13 $\frac{1}{\sinh x\cosh x}$ 14 $(\cosh x)e^{\sinh x}$

15 $\frac{\cosh x - x\sinh x}{\cosh^2 x}$ 16 $\frac{x\sinh x - \cosh x}{x^2}$

17 $3\cosh^2 x\sinh x\, e^{\cosh^3 x}$

18 $-\frac{2x\operatorname{cosech}^2 2x + 3\coth 2x}{x^4}$

19 $\frac{-\operatorname{cosech}(x^2)(2x^2\coth x^2 + 1)}{x^2}$

20 $\frac{\operatorname{sech} x(\operatorname{sech} x + \tanh x)}{\tanh x - \operatorname{sech} x}$ 21 $\frac{1}{\sqrt{2}}$

22 $y - \frac{9}{2} = -\frac{2}{7}(x - \ln 2)$

23 $(\ln 3, 0)$, $y = 3(x - \ln 3)$

24 $y = 5$ at $x = -\ln 5$

25 $\frac{c}{2}(3e^{-2} - e^2)$

26 max. $(-0.962, 11.18)$,
min. $(0.962, -11.18)$

29 $\frac{dy}{dx} = 3\sinh 3x\sin x + \cosh 3x\cos x$

$\frac{d^2y}{dx^2} = 8\cosh 3x\sin x + 6\sinh 3x\cos x$

30 $x - \frac{x^3}{3} + \dots$

31 $y - 0.905 = 0.181(x - 1.5)$
$y - 0.905 = -5.534(x - 1.5)$

32 $\frac{1}{\sqrt{(1+x^2)}}$ 33 $\frac{1}{\sqrt{(x^2-4)}}$ 34 $\frac{2x}{1-4x^2}$

35 $\frac{-1}{x\sqrt{(1-x^2)}}$ 36 $\frac{-1}{x\sqrt{(1+x^2)}}$ 37 $\frac{2}{1-4x^2}$

38 $\frac{-1}{2x\sqrt{(1-x)}}$ 39 $\operatorname{arcosh} x + \frac{x}{\sqrt{(x^2-1)}}$

40 $\frac{\operatorname{arsinh} x - \frac{x}{\sqrt{(1+x^2)}}}{(\operatorname{arsinh} x)^2}$

41 $\frac{2\operatorname{artanh} x}{1-x^2}$

42 $\frac{-1}{2x\sqrt{(1-x^2)}\sqrt{(\operatorname{arsech} x)}}$

43 $2xe^{x^2}\operatorname{arsinh} x + \frac{e^{x^2}}{\sqrt{(1+x^2)}}$

44 $\frac{\frac{1}{x}\operatorname{arcosh} x - \frac{\ln x}{\sqrt{(x^2-1)}}}{(\operatorname{arcosh} x)^2}$

45 $\frac{\cos x}{1-\sin^2 x} = \sec x$ 46 $\frac{\cosh x}{1-\sinh^2 x}$

47 $\frac{\operatorname{arsinh} x\sqrt{(1+x^2)} - \arcsin x\sqrt{(1-x^2)}}{(\operatorname{arsinh} x)^2\sqrt{(1-x^4)}}$

48 $y - x = 0$, $y - \operatorname{arsinh} 1 = \frac{1}{\sqrt{2}}(x - 1)$

50 $y - \frac{1}{2}\ln 7 = -\frac{7}{16}(x - \frac{3}{4})$

52 $(0, 3)$ 55 ±0.31 56 $(\ln 2, 4)$

57 $-\frac{1}{2}\operatorname{sech} x\tanh x$

60 (a) $(-1, 0)$ (b) $y + x + 1 = 0$