A-Level Further Maths

A21 Notes

Topics

1.1 Exponential form of complex numbers

You can use the modulus–argument form of a complex number to express it in the **exponential form**: $z = re^{i\theta}$.

Links The **modulus–argument** form of a complex number is $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$.
← Book 1, Section 2.3

You can write $\cos\theta$ and $\sin\theta$ as infinite series of powers of θ:

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots + \frac{(-1)^r\theta^{2r}}{(2r)!} + \ldots \quad (1)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots + \frac{(-1)^r\theta^{2r+1}}{(2r+1)!} + \ldots \quad (2)$$

You can also write e^x, $x \in \mathbb{R}$, as a series expansion in powers of x.

Links These are the Maclaurin series expansions of $\sin\theta$, $\cos\theta$ and e^x.
→ Chapter 2

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^r}{r!} + \ldots$$

You can use this expansion to define the exponential function for complex powers, by replacing x with a complex number. In particular, if you replace x with the imaginary number $i\theta$, you get

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \ldots$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} + \ldots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$$

By comparing this series expansion with (1) and (2), you can write $e^{i\theta}$ as

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula is known as **Euler's relation**. It is important for you to remember this result.

- **You can use Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$, to write a complex number z in exponential form:**

  $z = re^{i\theta}$

  where $r = |z|$ and $\theta = \arg z$.

Note Substituting $\theta = \pi$ into Euler's relation yields **Euler's identity**:

$$e^{i\pi} + 1 = 0$$

This equation links the five fundamental constants 0, 1, π, e and i, and is considered an example of mathematical beauty.

Example 1

Express the following in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.

a $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$ **b** $z = 5\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

Example 2

Express $z = 2 - 3i$ in the form $re^{i\theta}$, where $-\pi < \theta \leqslant \pi$.

Example 3

Express $z = \sqrt{2}e^{\frac{3\pi i}{4}}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

Example 4

Express $z = 2e^{\frac{23\pi i}{5}}$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

Example 5

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to show that $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$.

1.2 Multiplying and dividing complex numbers

You can apply the modulus–argument rules for multiplying and dividing complex numbers to numbers written in exponential form.

Recall that, for any two complex numbers z_1 and z_2,

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Links These results can be proved by considering the numbers z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$ and using the addition formulae for cos and sin. ← **Book 1, Section 2.3**

Applying these results to numbers in exponential form gives the following result:

■ **If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then:**

- $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Watch out You cannot automatically assume the laws of indices work the same way with complex numbers as with real numbers. This result only shows that they can be applied in these specific cases.

Example 7

Express $\dfrac{2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$ in the form $re^{i\theta}$.

1.3 De Moivre's theorem

You can use Euler's relation to find powers of complex numbers given in modulus–argument form.

$$\begin{aligned}(r(\cos\theta + i\sin\theta))^2 &= (re^{i\theta})^2\\ &= re^{i\theta} \times re^{i\theta}\\ &= r^2 e^{i2\theta}\\ &= r^2(\cos 2\theta + i\sin 2\theta)\end{aligned}$$

Similarly, $(r(\cos\theta + i\sin\theta))^3 = r^3(\cos 3\theta + i\sin 3\theta)$, and so on.

The generalisation of this result is known as **de Moivre's theorem**:

■ **For any integer n,**

$$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$$

You can prove de Moivre's theorem quickly using Euler's relation.

$$(r(\cos\theta + i\sin\theta))^n = (re^{i\theta})^n$$
$$= r^n e^{in\theta}$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

This step is valid for any integer exponent n. ← **Exercise 1B, Challenge**

You can also prove de Moivre's theorem for **positive integer exponents** directly from the modulus–argument form of a complex number using the addition formulae for sin and cos.

Links This proof uses the method of proof by induction. ← **Book 1, Chapter 8**

1. Basis step

$n = 1$; LHS $= (r(\cos\theta + i\sin\theta))^1 = r(\cos\theta + i\sin\theta)$

RHS $= r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$

As LHS = RHS, de Moivre's theorem is true for $n = 1$.

2. Assumption step

Assume that de Moivre's theorem is true for $n = k, k \in \mathbb{Z}^+$:

$(r(\cos\theta + i\sin\theta))^k = r^k(\cos k\theta + i\sin k\theta)$

3. Inductive step

When $n = k + 1$,

$$(r(\cos\theta + i\sin\theta))^{k+1} = (r(\cos\theta + i\sin\theta))^k \times r(\cos\theta + i\sin\theta)$$
$$= r^k(\cos k\theta + i\sin k\theta) \times r(\cos\theta + i\sin\theta)$$ — By assumption step
$$= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$
$$= r^{k+1}((\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta))$$
$$= r^{k+1}(\cos(k\theta + \theta) + i\sin(k\theta + \theta))$$ — By addition formulae
$$= r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta))$$

Therefore, de Moivre's theorem is true when $n = k + 1$.

4. Conclusion step

If de Moivre's theorem is true for $n = k$, then it has been shown to be true for $n = k + 1$.

As de Moivre's theorem is true for $n = 1$, it is now proven to be true for all $n \in \mathbb{Z}^+$ by mathematical induction.

Links The corresponding proof for negative integer exponents is left as an exercise. → **Exercise 1C, Challenge**

Example 8

Simplify $\dfrac{\left(\cos\frac{9\pi}{17} + i\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17} - i\sin\frac{2\pi}{17}\right)^3}$

Example 10

Use de Moivre's theorem to show that

$$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

You can also find trigonometric identities for $\sin^n\theta$ and $\cos^n\theta$ where n is a positive integer.

If $z = \cos\theta + \mathrm{i}\sin\theta$, then

$$\frac{1}{z} = z^{-1} = (\cos\theta + \mathrm{i}\sin\theta)^{-1}$$

$$= (\cos(-\theta) + \mathrm{i}\sin(-\theta))$$ — Apply de Moivre's theorem.

$$= \cos\theta - \mathrm{i}\sin\theta$$ — Use $\cos\theta = \cos(-\theta)$ and $-\sin\theta = \sin(-\theta)$.

It follows that

$$z + \frac{1}{z} = \cos\theta + \mathrm{i}\sin\theta + \cos\theta - \mathrm{i}\sin\theta = 2\cos\theta$$

$$z - \frac{1}{z} = \cos\theta + \mathrm{i}\sin\theta - (\cos\theta - \mathrm{i}\sin\theta) = 2\mathrm{i}\sin\theta$$

Also,

$z^n = (\cos\theta + \text{i}\sin\theta)^n = \cos n\theta + \text{i}\sin n\theta$ — By de Moivre's theorem.

$\frac{1}{z^n} = z^{-n} = (\cos\theta + \text{i}\sin\theta)^{-n}$

$= (\cos(-n\theta) + \text{i}\sin(-n\theta))$ — Apply de Moivre's theorem.

$= \cos n\theta - \text{i}\sin n\theta$ — Use $\cos\theta = \cos(-\theta)$ and $\sin(-\theta) = -\sin\theta$.

It follows that

$$z^n + \frac{1}{z^n} = \cos n\theta + \text{i}\sin n\theta + \cos n\theta - \text{i}\sin n\theta = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = \cos n\theta + \text{i}\sin n\theta - (\cos n\theta - \text{i}\sin n\theta) = 2\text{i}\sin n\theta$$

It is important that you remember and are able to apply these results:

- $z + \frac{1}{z} = 2\cos\theta$
- $z^n + \frac{1}{z^n} = 2\cos n\theta$
- $z - \frac{1}{z} = 2\text{i}\sin\theta$
- $z^n - \frac{1}{z^n} = 2\text{i}\sin n\theta$

Notation In exponential form, these results are equivalent to:

$\cos n\theta = \frac{1}{2}(\text{e}^{\text{i}n\theta} + \text{e}^{-\text{i}n\theta})$ $\quad$ $\sin n\theta = \frac{1}{2\text{i}}(\text{e}^{\text{i}n\theta} - \text{e}^{-\text{i}n\theta})$.

Example 11

Express $\cos^5\theta$ in the form $a\cos 5\theta + b\cos 3\theta + c\cos\theta$, where a, b and c are constants.

Example 12

a Express $\sin^4\theta$ in the form $d\cos 4\theta + e\cos 2\theta + f$, where d, e and f are constants.

b Hence find the exact value of $\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$.

1.6 nth roots of a complex number

You can use de Moivre's theorem to solve an equation of the form $z^n = w$, where $z, w \in \mathbb{C}$.
This is equivalent to finding the nth roots of w.

Just as a real number, x, has two square roots, $\sqrt{x}$ and $-\sqrt{x}$, any complex number has n distinct nth roots.

- **If z and w are non-zero complex numbers and n is a positive integer, then the equation $z^n = w$ has n distinct solutions.**

You can find the solutions to $z^n = w$ using de Moivre's theorem, and by considering the fact that the argument of a complex number is not unique.

Note $\cos(\theta + 2k\pi) = \cos\theta$ and $\sin(\theta + 2k\pi) = \sin\theta$ for integer values of k.

- **For any complex number $z = r(\cos\theta + i\sin\theta)$, you can write $z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$, where k is any integer.**

Example 15

a Solve the equation $z^3 = 1$.

b Represent your solutions to part **a** on an Argand diagram.

c Show that the three cube roots of 1 can be written as 1, ω and ω^2 where $1 + \omega + \omega^2 = 0$.

- **In general, the solutions to $z^n = 1$ are $z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) = e^{\frac{2\pi ik}{n}}$ for $k = 1, 2, \ldots, n$ and are known as the nth roots of unity.**

  If n is a positive integer, then there is an nth root of unity $\omega = e^{\frac{2\pi i}{n}}$ such that:

  - **the nth roots of unity are $1, \omega, \omega^2, \cdots, \omega^{n-1}$**
  - **$1, \omega, \omega^2, \cdots, \omega^{n-1}$ form the vertices of a regular n-gon**
  - **$1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$**

Example 16

Solve the equation $z^4 = 2 + 2i\sqrt{3}$.

Example 17

Solve the equation $z^3 + 4\sqrt{2} + 4i\sqrt{2} = 0$.

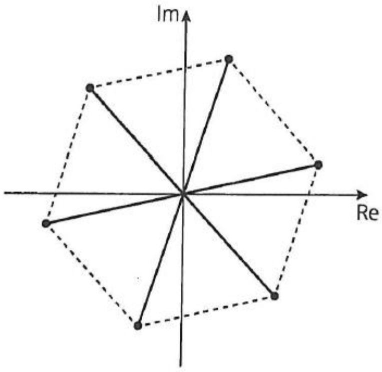
1.7 Solving geometric problems

You can use properties of complex nth roots to solve geometric problems.

- **The nth roots of any complex number a lie at the vertices of a regular n-gon with its centre at the origin.**

The orientation and size of the regular polygon will depend on a.

Notation The centre of a regular polygon is considered to be the centre of the circle that passes through all of its vertices.



For example, the sixth roots of 7 + 24i form this regular hexagon. Each vertex of the hexagon is equidistant from the origin, which lies at the centre of the circle passing through all six vertices.

Online Explore nth roots of complex numbers in an Argand diagram using GeoGebra.

You can find the vertices of this regular polygon by finding a single vertex, and rotating that point around the origin. This is equivalent to multiplying by the nth roots of unity.

- **If z_1 is one root of the equation $z^n = s$, and $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the nth roots of unity, then the roots of $z^n = s$ are given by $z_1, z_1\omega, z_1\omega^2, \ldots, z_1\omega^{n-1}$.**

Example 18

The point $P(\sqrt{3}, 1)$ lies at one vertex of an equilateral triangle. The centre of the triangle is at the origin.

a Find the coordinates of the other vertices of the triangle.

b Find the area of the triangle.

Summary of key points

1 You can use **Euler's relation**, $e^{i\theta} = \cos\theta + i\sin\theta$, to write a complex number z in exponential form:

$$z = re^{i\theta}$$

where $r = |z|$ and $\theta = \arg z$.

2 For any two complex numbers $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$,

- $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
- $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

3 **De Moivre's theorem:**

For any integer n, $(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

4
- $z + \dfrac{1}{z} = 2\cos\theta$
- $z^n + \dfrac{1}{z^n} = 2\cos n\theta$
- $z - \dfrac{1}{z} = 2i\sin\theta$
- $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$

5 For $w, z \in \mathbb{C}$,

- $\displaystyle\sum_{r=0}^{n-1} wz^r = w + wz + wz^2 + \ldots + wz^{n-1} = \frac{w(z^n - 1)}{z - 1}$
- $\displaystyle\sum_{r=0}^{\infty} wz^r = w + wz + wz^2 + \ldots = \frac{w}{1 - z}, |z| < 1$

6 If z and w are non-zero complex numbers and n is a positive integer, then the equation $z^n = w$ has n distinct solutions.

7 For any complex number $z = r(\cos\theta + i\sin\theta)$, you can write

$$z = r(\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))$$

where k is any integer.

8 In general, the solutions to $z^n = 1$ are $z = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) = e^{\frac{2\pi ik}{n}}$ for $k = 1, 2, \ldots, n$ and are known as the nth roots of unity.

If n is a positive integer, then there is an nth root of unity $\omega = e^{\frac{2\pi i}{n}}$ such that:

- The nth roots of unity are $1, \omega, \omega^2, \ldots, \omega^{n-1}$
- $1, \omega, \omega^2, \ldots, \omega^{n-1}$ form the vertices of a regular n-gon
- $1 + \omega + \omega^2 + \ldots + \omega^{n-1} = 0$

9 The nth roots of any complex number s lie on the vertices of a regular n-gon with its centre at the origin.

10 If z_1 is one root of the equation $z^n = s$, and $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the nth roots of unity, then the roots of $z^n = s$ are given by $z_1, z_1\omega, z_1\omega^2, \ldots, z_1\omega^{n-1}$.

A21 Further Maths

Partial Fractions

*knowledge of A-level Maths A21 Partial Fractions is assumed.

The process of taking a single fraction and breaking it up into the sum (or difference) of 2 or more fractions is known as splitting an expression into partial fractions.

Note: If the degree of the numerator is greater than or equal to the degree of the denoinator you must first divide the numerator by the denominator.

Quadratic factors in the denominator

For a fraction that has a non-reducible quadratic factor on the denominator and where the degree of the denominator exceeds that of the numerator e.g.

$$\frac{x^2 - 5x + 1}{(x^2 + 1)(x - 2)}$$

The partial fractions are of the form:-

$\frac{Ax+B}{(x^2+1)} + \frac{C}{(x-2)}$ where A,B and C are constants.

Example

Express $\frac{5x^2+4x+4}{(x+2)(x^2+4)}$ in partial fractions.

Solution

Note:- Remember to check that the denominator is completely factorised before attempting to put in partial fractions.

Example

Express $\frac{-2x-1}{(x^2-3x+2)(x^2-x+3)}$ in partial fractions.

Solution

Note:-

$$x^3 - 1 = (x-1)(x^2+x+1)$$

$$x^3 + 1 = (x+1)(x^2-x+1)$$

$$x^3 - y^3 = (x-y)(x^2+xy+y^2)$$

$$x^3 + y^3 = (x+y)(x^2-xy+y^2)$$

*P2 book Ex1D Q6,7,8,9,10,12,23,24,28,29,31,33,35

Summation of Finite Series Using The Method Of Differences

$$\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n \ (frontwards)$$

$$\sum_{r=1}^{n} r = n + (n-1) + (n-2) \ldots 3 + 2 + 1 \ (backwards)$$

Adding:-

$$2\sum_{r=1}^{n} r = (n+1) + (n+1) + (n+1) + \cdots (n+1) + (n+1) + (n+1)$$

(n terms)

$$2\sum_{r=1}^{n} r = n(n+1)$$

Result 1:-

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Note:- Here is another way you could sum the series $1 + 2 + 3 + \cdots + n$.

Consider the identity

$$2r \equiv r(r+1) - (r-1)r$$

Taking successive values 1,2,3,….,n for r, we get:-

This method is called summing a series by the method of difference.

Genereally if it is possible to find a function f® such that the rth term u_r of a series can be expressed as

$u_r = f(r+1) - f(r)$, then it is easy to find

$$\sum_{r=1}^{n} u_r$$

We have for r=1,2,3,…,n

$$u_1 = f(2) - f(1)$$

$$u_2 = f(3) - f(2)$$

$$u_3 = f(4) - f(3)$$

.. .. ..

.. .. ..

$$u_n = f(n+1) - f(n)$$

Adding:-

$$\sum_{r=1}^{n} u_r = f(n+1) - f(1)$$

because all the other terms on R.H.S. cancel out.

Example 1:-Find

$$\sum_{r=1}^{n} r^2$$

Consider the identity

$$24r^2 + 2 \equiv (2r+1)^3 - (2r-1)^3$$

And take r=1,2,3,…,n.

Solution

Result 2:-

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Example 2:-Find

$$\sum_{r=1}^{n} r^3$$

Consider the identity

$$4r^3 \equiv r^2(r+1)^2 - (r-1)^2 r^2$$

And take r=1,2,3,…,n.

Solution

Result 3:-

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Note:- Since

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

Then

$$\sum_{r=1}^{n} r^3 = \left(\sum_{r=1}^{n} r\right)^2$$

Example 3:- Find

$$\sum_{r=1}^{n} r(r+1)$$

Consider the identity

$$3r(r+1) \equiv r(r+1)(r+2) - (r-1)(r)(r+1)$$

And take r=1,2,3,…,n.

Solution

Results for the sigma notation:-

1.

$$\sum_{r=1}^{n} af(r) = a\sum_{r=1}^{n} f(r)$$

Proof:-

$$\sum_{r=1}^{n} af(r) = af(1) + af(2) + af(3) + \cdots + af(n)$$

$$\sum_{r=1}^{n} af(r) = a[f(1) + f(2) + f(3) + \cdots + f(n)]$$

$$\therefore \sum_{r=1}^{n} af(r) = a\sum_{r=1}^{n} f(r)$$

2.

$$\sum_{r=1}^{n} f(r) + g(r) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$$

Proof:-

$$\sum_{r=1}^{n} f(r) + g(r) = f(1) + g(1) + f(2) + g(2) + \cdots f(n) + g(n)$$

$$\sum_{r=1}^{n} f(r) + g(r) = [f(1) + f(2) + \cdots + f(n)] + [g(1) + g(2) + \cdots + g(n)]$$

$$\sum_{r=1}^{n} f(r) + g(r) = \sum_{r=1}^{n} f(r) + \sum_{r=1}^{n} g(r)$$

Questions: P3 book Page 15 Exercise 2A Q1,3,4,7,9,10

Telescoping Series

Example:- Find the value of

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

Solution

The above is an example of a telescoping series, since the terms of S_n, other than the first and last, cancel out in pairs.

Summation of Finite Series Using Standard Results

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

Example:- Find

$(a.)\sum_{r=7}^{20} r^2$

$(b.)\sum_{r=12}^{25} r^3$

Solution

Example:- Show

$$\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$$

Solution

Example:- Find the following in terms of n.

$$\sum_{r=1}^{n} 6r^2 + 2^r$$

Solution

***Questions P3 book page 18 Ex2B Q1-3,5,-7,9-11,13,14,17,19

A-level Further Maths A21

Induction

A theorem thought to be true for all values of the positive integer n, can be proved by showing that:-

(i) If it is true for n=k, then it is also true for n=k+1.

and

(ii) It is true for some small value of n such as n=1 (or perhaps n=2 or n=3)

If you prove both (i) and (ii) then you have shown that the theorem is true at the start (usually n=1) and it ids true for n=1+1 and n=2+1 and n=3+1 and so on for all integer values of n following on after the valid starting value (usually n=1).

Example 1 Use the method of mathematical induction to prove:-

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

where n is a positive integer.

Proof

Example 2 Use the method of mathematical induction to prove that the expression:-

$$3^{2n} + 7$$

Is divisible by 8 for all positive integers n.

Proof (Method 1)

**See other method too

Proof (Method 2)

Example Given that n is an integer, which is greater than 3, show that

$$n! > 2^n$$

Proof

**P4 Book Page 279 Ex8A Q1-6,9,17,20,25,29,30,34, Extras: Q8,13,15

Maclaurins Series

Let f(x) be a function, which throughout a certain domain, including x=0 is

(a.) Differentiable any number of times ,and
(b.) The sum of a convergent power series.

Let this series be

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$so\ f(0) = a_0$$

*differentiating term by term and putting x=0

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$$

$$so\ f'(0) = a_1$$

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots$$

$$so\ f''(0) = 2a_2$$

$$or\ f''(0) = 2!\,a_2$$

$$f'''(x) = 6a_3 + 24a_4x + 60a_5x^2 + \cdots$$

$$so\ f'''(0) = 6a_3$$

$$or\ f'''(0) = 3!\,a_2$$

$$so\ you\ could\ write\ f(x) = f(0) + xf'(0) + \frac{x^2f''(0)}{2!} + \frac{x^3f'''(0)}{3!} + \frac{x^4f^{\prime v}(0)}{4!} + \cdots + \frac{x^nf^n(0)}{n!} + \cdots$$

This is Maclaurins Series.

Exponential Series

$Let\ f(x) = e^x$

Logarithmic Series

$Let\ f(x) = \ln(1+x)$

Example

Expand $\cos x$ in ascending powers of x.

Solution

***P3 Book Exercise 2D

(next bit is not needed. Just to show)

Challenge

The **ratio test** is a sufficient condition for the convergence of an infinite series. It says that a series $\sum_{r=1}^{\infty} a_r$ converges if $\lim_{r \to \infty}\left|\frac{a_{r+1}}{a_r}\right| < 1$, and diverges if $\lim_{r \to \infty}\left|\frac{a_{r+1}}{a_r}\right| > 1$.

Use the ratio test to show that

a the Maclaurin series expansion of e^x converges for all $x \in \mathbb{R}$

Problem-solving

If $\lim_{r \to \infty}\left|\frac{a_{r+1}}{a_r}\right| = 1$ or does not exist then the ratio test is inconclusive.

$$e^x = \sum_{r=1}^{\infty} \frac{x^r}{r!}$$

$$= \lim_{r \to \infty}\left|\frac{\frac{x^{r+1}}{(r+1)!}}{\frac{x^r}{r!}}\right| = \lim_{r \to \infty}\left|\frac{x}{(r+1)}\right|$$ which is <1 for all x.

Binomial Series

Consider $f(x) = (1+x)^n \quad for\, n \in R$

$$f(x) = (1+x)^n \quad so\ f(0) = 1$$

$$f'(x) = n(1+x)^{n-1} \quad so\ f'(0) = n$$

$$f''(x) = n(n-1)(1+x)^{n-2} \quad so\ f''(0) = n(n-1)$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \quad so\ f'''(0) = n(n-1)(n-2)$$

.

.

.

$$f^r(x) = n(n-1)(n-2)..(n-r+1)(1+x)^r \quad so\ f^r(0) = n(n-1)(n-2)..(n-r+1)$$

Maclaurins gives:-

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + .. + \frac{n(n-1)(n-2)..(n-r+1)x^r}{r!}$$

Which is the Binomial Series for any $n \in R$ and is convergent, provided $|x| < 1$.

If $n \in Z^+$, the series terminates and reduces to the Binomial Theorem.

<u>Note</u> Define $\binom{n}{r}$ to be

$$\binom{n}{r} = \frac{n(n-1)(n-2)..(n-r+1)}{r!}$$

<u>Example</u>

Expand $(1-3x)^{\frac{-2}{3}}$ up to terms including x^3.

<u>Solution</u>

Example

Expand $(1-3x)^{\frac{1}{5}}$ inascending powers of x up to the term x^3. Take $x=\frac{1}{32}$ to find an approprimation for $29^{\frac{1}{5}}$, giving your answer correct to 5d.p.

Solution

Example

$$f(x)=\frac{x}{(3-2x)(2-x)}$$

(a.) Express $f(x)$ in partial fractions
(b.) Expand $f(x)$ up to terms including x^3.
(c.) State the set of values of x for which the series is valid.

Solution

**Exercise 2C Q17,20-25

Using the Polynomial Series Form of Functions To Find Approximations For The Functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

So if you take terms in x^3 and higher powers of x to be negligible, then

$\sin x \approx x$ and $\cos x \approx 1 - \frac{x^2}{2}$ where x is small.

Also

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

So for small x, $\tan x \approx x$.

Example

Find a quadratic polynomial approximation for $\frac{\sin 2x}{1+x}$, give that x is small.

Solution

Example

Given that x is small, show that $\frac{3\sin x}{2+\cos x} \approx x$.

Solution

Example

Show that $\lim_{x \to 0} \frac{1 - \cos 4x + x \sin 3x}{x^2} = 11$

Solution

**pg 29 Exercise 2E (use the standard Maclaurins results in the formula booklet)

What is an improper integral?

An improper integral is a definite integral for which the integrand (the expression to be integrated) is undefined either within at one or both of the limits of integration, or at some point between the limits of integration.

For example:

$\int_1^{\infty} \frac{1}{x^2}\,dx$ is an improper integral since one of its limits is infinity;

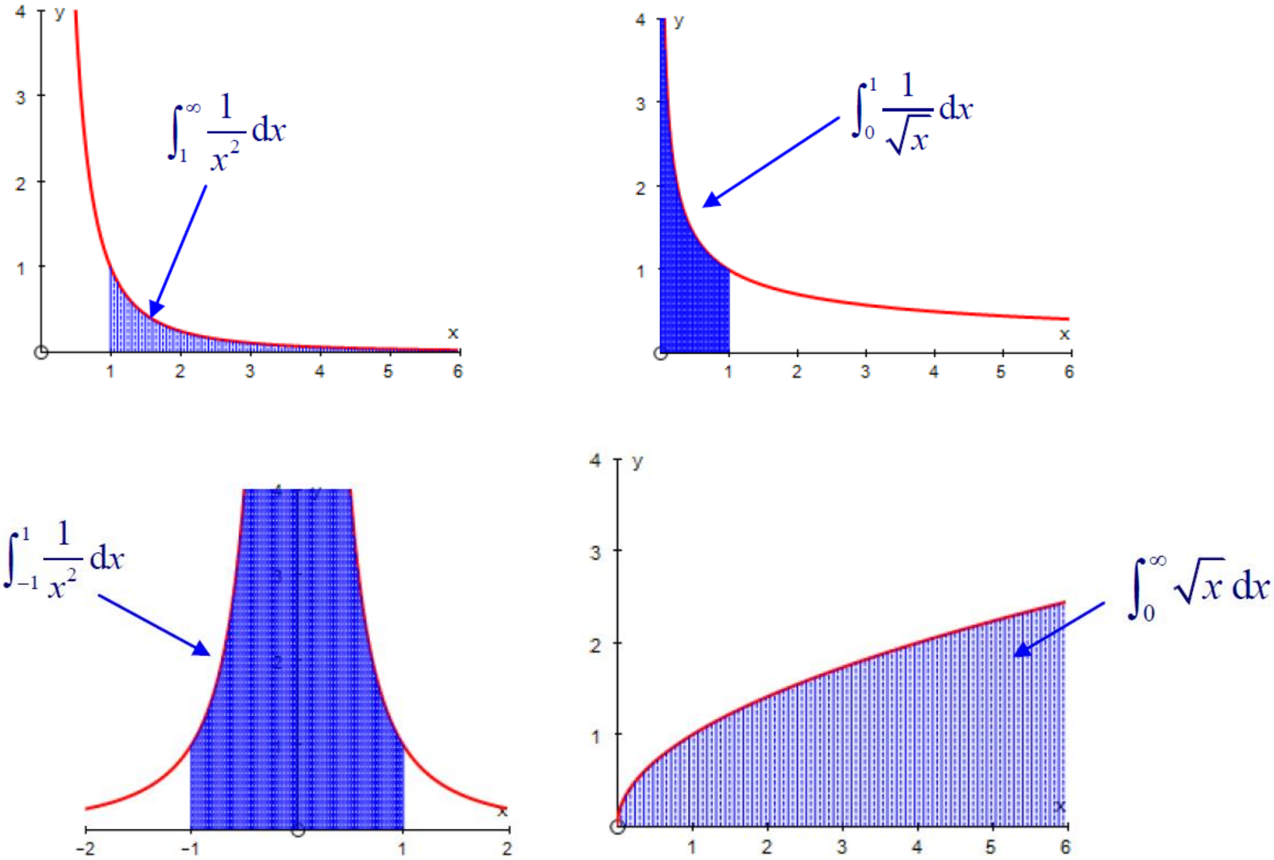
$\int_0^{1} \frac{1}{\sqrt{x}}\,dx$ is an improper integral since it is undefined at $x = 0$.

$\int_{-1}^{1} \frac{1}{x^2}\,dx$ is an improper integral since it is undefined at $x = 0$;

$\int_0^{\infty} \sqrt{x}\,dx$ is an improper integral since one of its limits is infinity;

Some improper integrals can be evaluated, others cannot. Remember that definite integration is equivalent to finding the area under a graph between two points. In some cases, an improper integral represents a finite area, even though the integrand is undefined at some point.

These are the graphs representing each of the four improper integrals above:



It is clear that the area in the fourth graph is infinite, and therefore that the value of the integral is undefined. However, in the cases of the other three graphs, the graphs are all approaching one of the axes, so it is possible that the area under the graphs may be finite, and therefore that the integral can be evaluated.

You can decide whether or not an improper integral has a finite value, and if so, what it is, by considering limits.

Improper integrals with limits involving infinity

For improper integrals with limits involving infinity, you replace the limit of infinity with a variable, work out the value of the integral in terms of the variable, and then look at what happens as the value of the variable tends to infinity.

Example 1

Find, if possible, the values of

(i) $\int_1^{\infty} \frac{1}{x^2} \, dx$

(ii) $\int_0^{\infty} \sqrt{x} \, dx$

Solution

(i)

(ii)

As expected from the graph, the integral $\int_0^{\infty} \sqrt{x} \, dx$ cannot be found.

However, the integral $\int_1^{\infty} \frac{1}{x^2} \, dx$ has been shown to have a finite value.

Improper integrals where the integral is undefined at a particular value

For improper integrals where the integrand is undefined at one of the limits of integration, you use a similar technique to the one above: you replace the limit with a variable, work out the value of the integral in terms of the variable, and then look at what happens as the value of the variable tends to the original value.

If the integrand is undefined at a point between the limits, you need to split the integral into two parts, so that the problem value is a limit of both parts, and then use the technique above.

Example 2

Find, if possible, the values of

(i) $\int_0^1 \frac{1}{\sqrt{x}} \, dx$

(ii) $\int_{-1}^1 \frac{1}{x^2} \, dx$

Solution

(i)

(ii)

A21 Further Maths

Differentiation and Integration of Inverse Trig Functions

Graphs of inverse trigonometric functions

$y = \arcsin x$

Remember we defined

$$y = \sin x$$

to have domain

$$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

and the range is

$$-1 \leqslant \sin x \leqslant 1$$

Our inverse

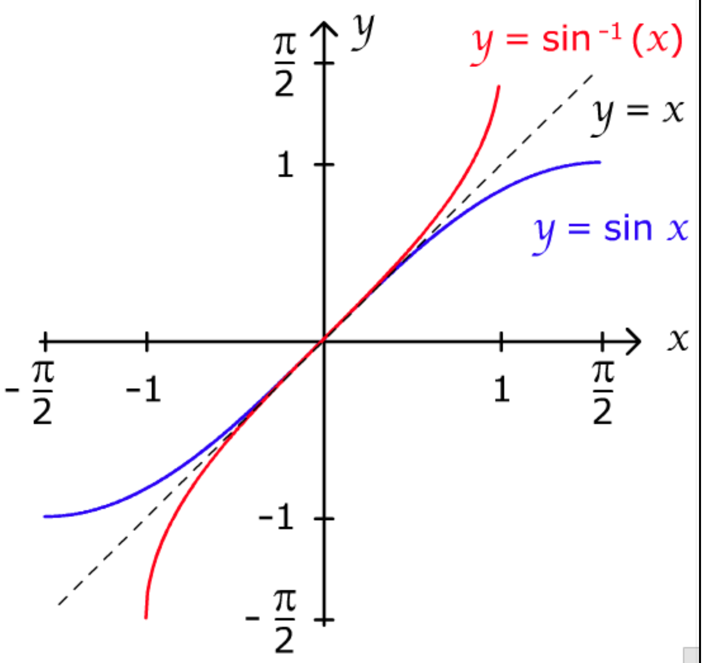
$$y = \sin^{-1} x$$

has domain

$$-1 \leqslant x \leqslant 1$$

and range

$$-\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$$



$y = \arccos x$

We limit the domain to

$$0 \leqslant x \leqslant \pi$$

and the range is

$$-1 \leqslant \cos x \leqslant 1$$

The inverse function looks like this.

It's a reflection of $y = \cos x$ in the line $y = x$.

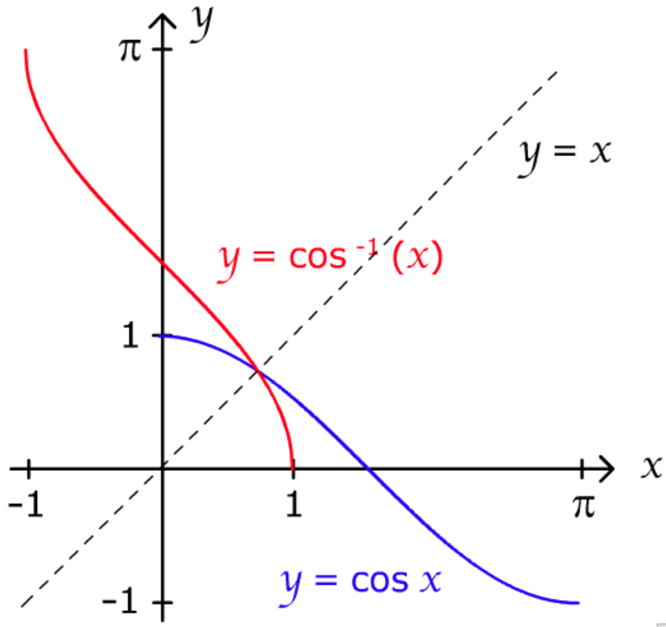
The domain of

$$y = \cos^{-1}(x)$$

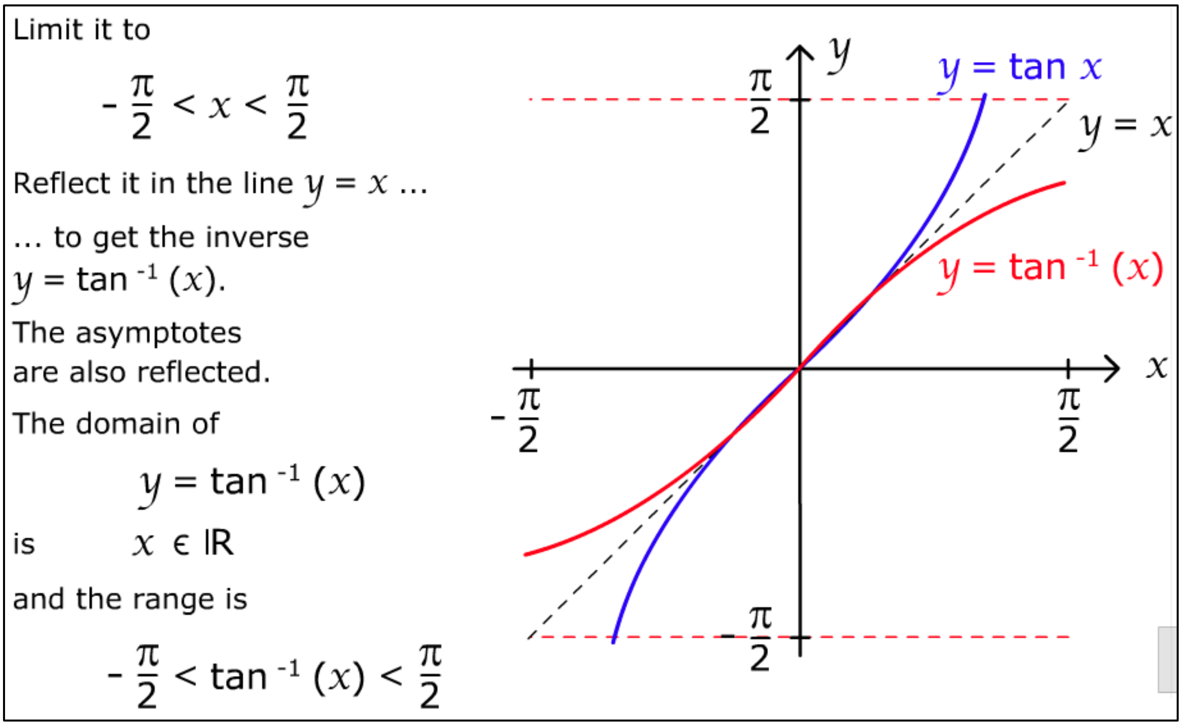
is $\quad -1 \leqslant x \leqslant 1$

and the range is

$$0 \leqslant \cos^{-1}(x) \leqslant \pi$$



$\underline{y = \arctan x}$



Differentiation of Inverse Trig Functions

1. Let $y = \sin^{-1} x \;\therefore\; x = \sin y \qquad -1 \le x \le 1$ and $\frac{-\pi}{2} \le y \le \frac{\pi}{2}$

$$\therefore \cos y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\pm\sqrt{1 - x^2}}$$

$$* \, But\; y = \sin^{-1} x \;\; is\; an\; increasing\; function\; between$$
$$-1 \; and \; 1, so \; \frac{dy}{dx} \; is\; positive.$$

$$\therefore \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

2. Let $y = \cos^{-1} x \;\therefore\; x = \cos y \qquad -1 \le x \le 1$ and $0 \le y \le \pi$

$$\therefore\; -\sin y \frac{dy}{dx} = 1$$

$$\therefore\; \frac{dy}{dx} = \frac{-1}{\sin y}$$

$$\therefore\; \frac{dy}{dx} = \frac{-1}{\pm\sqrt{1 - x^2}}$$

$* \, But$ $y = \cos^{-1} x$ $is \; a \; decreasing \; function \; between$ $-1 \; and \; 1, so \; \frac{dy}{dx} \; is \; negative.$

$$\therefore\; \frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1 - x^2}}$$

3. Let $y = \tan^{-1} x \;\therefore\; x = \tan y \; -\infty < x < \infty$ (or $x \in R$)and $\frac{-\pi}{2} < y < \frac{\pi}{2}$

$$\therefore\; \sec^2 y \frac{dy}{dx} = 1$$

$$\therefore\; \frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$* \sec^2 y = 1 + \tan^2 y$$

$$\therefore\; \frac{dy}{dx} = \frac{1}{1 + x^2}$$

$$\therefore\; \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1 + x^2}$$

Results

$$\frac{d(sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}},$$

$$\frac{d(tan^{-1} x)}{dx} = \frac{1}{1 + x^2}$$

Example Find $\frac{dy}{dx}$ when

(a.) $y = \cos^{-1} x^2$

(b.) $y = \tan^{-1}(e^{3x})$

Example Find an equation of the normal to the curve $y = \sin^{-1} 2x$ at point where $x = \frac{1}{4}$.

*P3 Book Ex3C Q7-15

Integration of $\frac{1}{a^2+x^2}$ **and** $\frac{1}{\sqrt{a^2-x^2}}$

1. Since $\frac{d(sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

then $\frac{d(sin^{-1}\frac{x}{a})}{dx} = \frac{1\times\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$

$\therefore \frac{d(sin^{-1}\frac{x}{a})}{dx} = \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}$ $\quad\therefore \frac{d(sin^{-1}\frac{x}{a})}{dx} = \frac{1}{\sqrt{a^2-x^2}}$

$hence \int \frac{1}{\sqrt{a^2-x^2}} dx = sin^{-1}\frac{x}{a} + c$

2. And since $\frac{d(tan^{-1} x)}{dx} = \frac{1}{1+x^2}$

then $\frac{d(tan^{-1}\frac{x}{a})}{dx} = \frac{1\times\frac{1}{a}}{1+\left(\frac{x}{a}\right)^2}$

$\therefore \frac{d(tan^{-1}\frac{x}{a})}{dx} = \frac{1}{a(1+\left(\frac{x}{a}\right)^2)}$

$\therefore \frac{d(tan^{-1}\frac{x}{a})}{dx} = \frac{1}{a\left(\frac{a^2+x^2}{a^2}\right)}$

$\therefore \frac{d(tan^{-1}\frac{x}{a})}{dx} = \frac{1}{a\left(\frac{a^2+x^2}{a^2}\right)}$

$\therefore \frac{d(tan^{-1}\frac{x}{a})}{dx} = \frac{1}{\left(\frac{a^2+x^2}{a}\right)}$ $\quad\therefore \frac{d(tan^{-1}\frac{x}{a})}{dx} = \frac{a}{a^2+x^2}$

$hence \int \frac{1}{a^2+x^2} dx = \frac{1}{a} tan^{-1}\frac{x}{a} + c$

Example Find (a.) $\int \frac{4}{x^2+16}\,dx$ (b.) $\int \frac{2}{36+x^2}\,dx$

Example Evaluate $\int_{-1.5}^{0} \frac{1}{\sqrt{9-x^2}}\,dx$

*UPM Ex15H Q13-24

Integration by Parts

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx} dx$$

Example Find $\int xe^{x}\, dx$.

Example Find $\int xcosx\, dx$.

Example Find $\int lnx\, dx$.

Example Find $\int x^2 e^{2x}\,dx$.

Example Find $I = \int e^x sinx\,dx$.

Example Find $I = \int_0^1 x(x-1)^3 \, dx$ (definite integral)

*P2 Book Ex9D Q1,3,5,6,9,14,15,16,18,19

Reduction Formula

Example If $I_n = \int x^n e^{-x}\, dx$ evaluate I_3.

Example

If $I_n = \int_0^{\frac{\pi}{2}} sin^n x\, dx$, show that $I_n = \frac{n-1}{n} I_{n-2}\ \ for\ n \geq 2.$

Hence find (a.) I_5 and (b.) I_6

Example Use the identity $sec^2 A \equiv 1 + tan^2 A$ to find a reduction formula for

$$I_n = \int_0^{\frac{\pi}{4}} tan^n x \, dx$$

Hence, evaluate (a.) $\int_0^{\frac{\pi}{4}} tan^5 x \, dx$ and (b.) $\int_0^{\frac{\pi}{4}} tan^6 x \, dx$

*P4 book Ex5A Q1,2,3,5,7,9-13,15, extras Q4,6 (tricky)

The exponential functions can be combined to form functions that have strong similarities to trig (or circular) functions. These functions are called hyperbolic cosine (cosh x) and hyperbolic sine (sinh x).

$\cosh x = \frac{e^x+e^{-x}}{2}\ for\ x \in R$ similar to $\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$\sinh x = \frac{e^x-e^{-x}}{2}\ for\ x \in R$ similar to $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

These two definitions are basic and from them four other hyperbolic functions are defined:-

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\therefore \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad for\ x \in R$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \qquad for\ x \in R$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \qquad for\ x \in R, x \neq 0$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^{2x} + 1}{e^{2x} - 1} \qquad for\ x \in R, x \neq 0$$

Graphs of Hyperbolic Functions

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^{x}}{2} = \frac{-(e^{x} - e^{-x})}{2} = -\sinh x$$

So $\sinh x$ is an odd function.

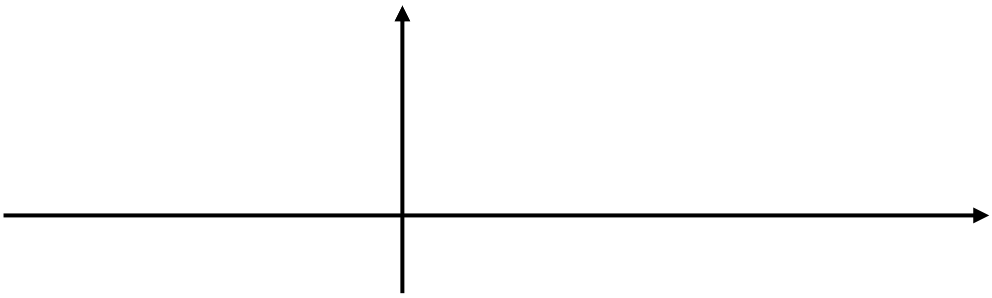
Similarly

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{x} + e^{-x}}{2} = \cosh x$$

So $\cosh x$ is an even function.

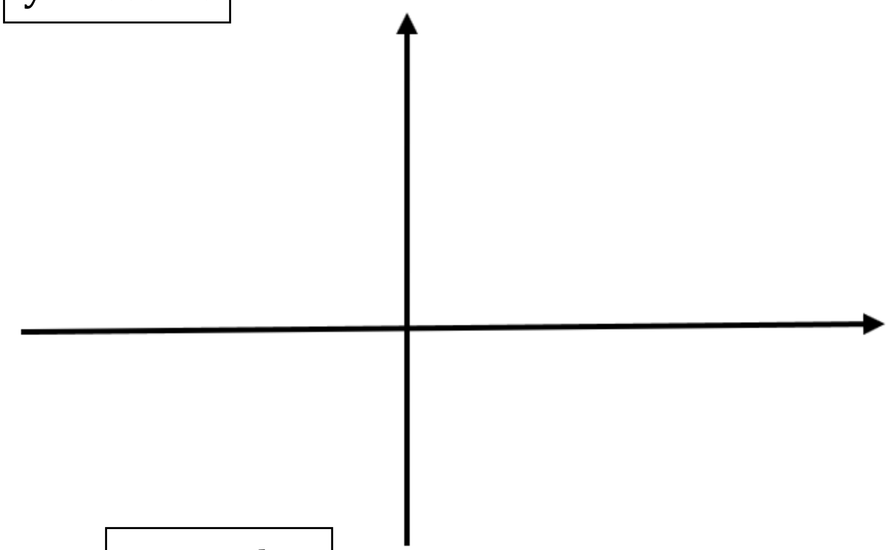
Also $\cosh x = \frac{e^{x}+e^{-x}}{2} > \frac{e^{x}-e^{-x}}{2} = \sinh x$ for all values.

First sketch $y = e^{x}\ and\ y = e^{-x}$



So

$\mathrm{y} = \cosh x$



$\mathrm{y} = \sinh x$

Since $\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$, we see at $x = 0, \tanh x = 0$.

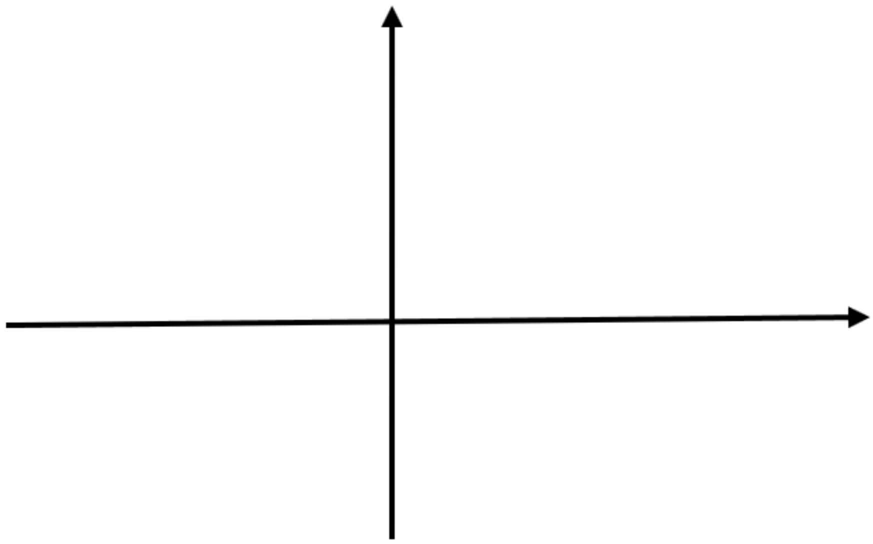
Also, $\tanh(-x) = \frac{e^{-2x}-1}{e^{-2x}+1} = \frac{\frac{1}{e^{2x}}-1}{\frac{1}{e^{2x}}+1} = \frac{1-e^{2x}}{1+e^{2x}} = -\tanh x$

So $\tanh x$ is an odd function.

Now $\tanh x = \frac{e^{2x}-1}{e^{2x}+1} = \frac{1-e^{-2x}}{1+e^{-2x}}$ (by dividing through by e^{2x})

As $x \to \infty, e^{-2x} \to 0$ and $\tanh x \to 1$

As $x \to -\infty, e^{2x} \to 0$ and $\tanh x \to -1$



The lines $y = \pm 1$ are asymptotes to the curve.

<u>Example</u> Sketch $y = \operatorname{sech} x \ \ for\ x \in R$.

Example Find the exact values of x for which $\tanh x = \frac{1}{2}$.

Identities

Example Prove $cosh^2 x - sinh^2 x \equiv 1$

Example Prove $\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y$

Example Find an identity for $\sinh 2A$ in terms of $\cosh A$ and $\sinh A$.Hence find an identity for $\tanh 2A$.

Osborne's Rule:- The formulae for circular and hyperbolic functions correspond exactly, provided the sign is changed whenever there exists a product (or implied product) of 2 sines.

i.e. the rule is to replace each trig function with its corresponding hyperbolic function and change the sign of every product (or implied product) of 2 sines.

e.g. $\cos 2A = 1 - sin^2 A$

becomes $\cosh 2A = 1 + sinh^2 A$

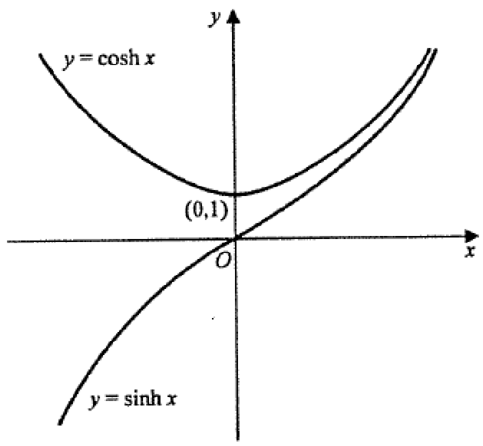
e.g. $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

becomes $\tan(A - B) = \frac{\tanh A - \tanh B}{1 - \tanh A \tanh B}$

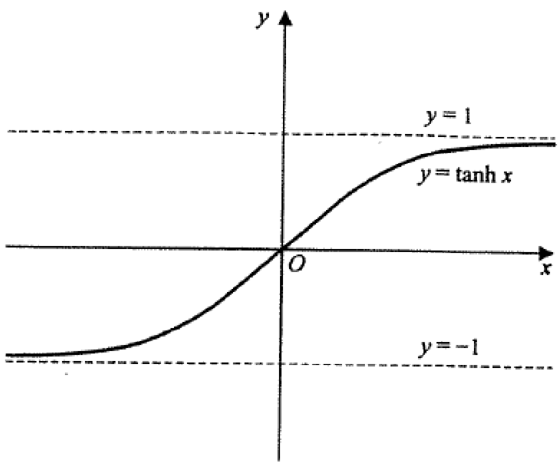
**P3 book Ex4A Q(1,2,3)alt parts, 4,5,7-17odds,18,20,22,23,25

Graphs of Hyperbolic Functions

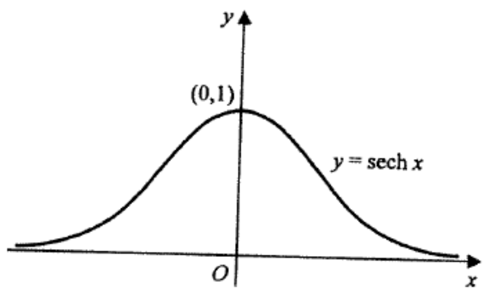
1.



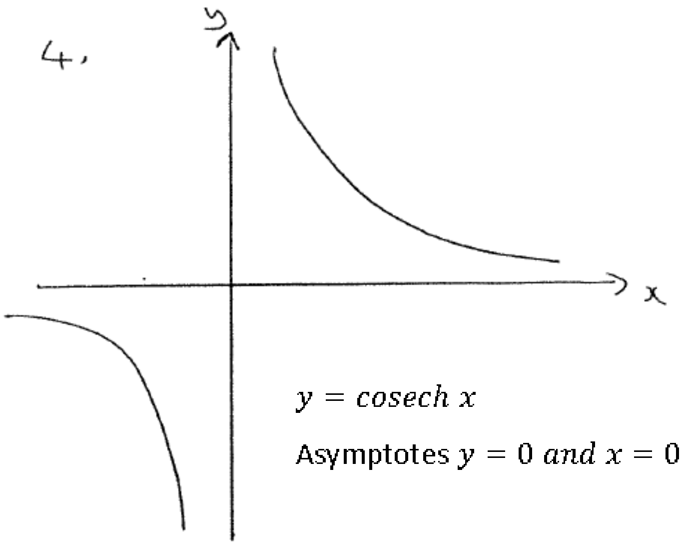
2.



3.



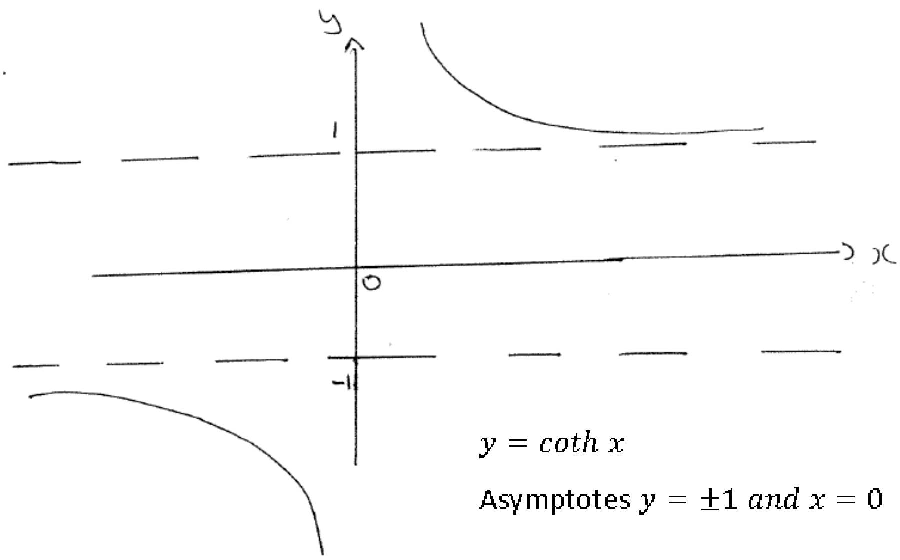
4.



$y = cosech\ x$

Asymptotes $y = 0\ and\ x = 0$

5.

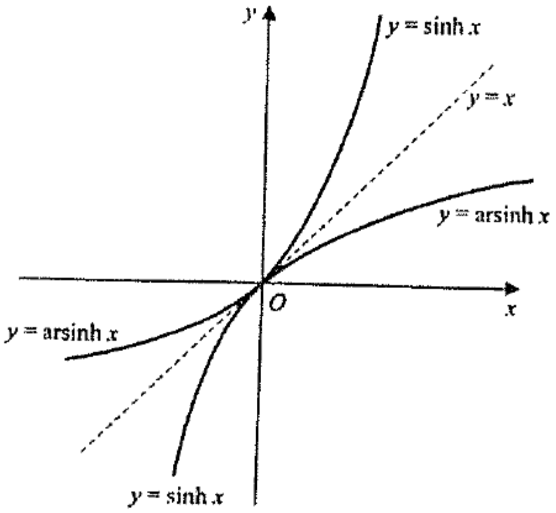


$y = coth\ x$

Asymptotes $y = \pm 1\ and\ x = 0$

Inverse Hyperbolic Functions

1.



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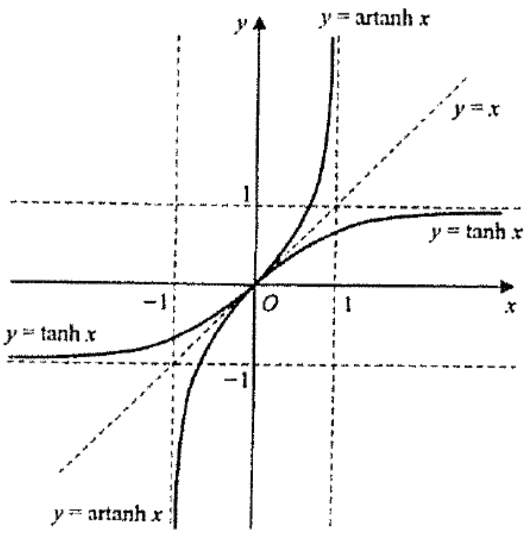
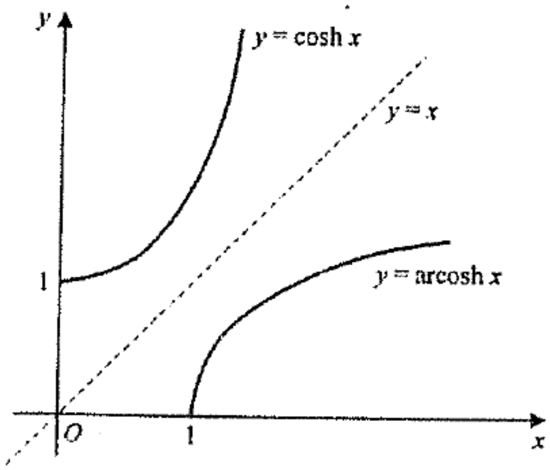


3.

For the function $\cosh x$, you need to take the domain $x \geqslant 0$, so that it is a one–one function. Then the inverse function **arcosh** $\boldsymbol{x}$ is defined for the domain $x \geqslant 1$ and range $\text{arcosh}\, x \geqslant 0$. The graphs of $\cosh x$ and $\text{arcosh}\, x$ look like this:

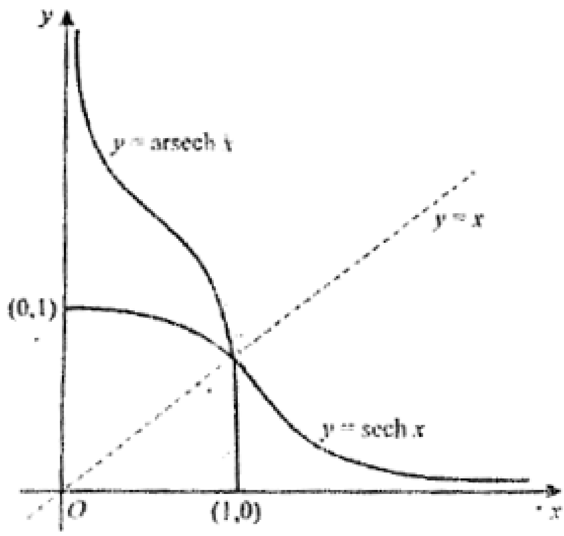


4.

In the same diagram, sketch the curves

$y = \text{sech}\, x,\ x \in \mathbb{R},\ x \geqslant 0$
$y = \text{arsech}\, x,\ x \in \mathbb{R},\ 0 < x \leqslant 1$

The curves are shown in the diagram. One is the reflection of the other in the line $y = x$.



The Logarithmic Form of Inverse Hyperbolic Functions

If $y = sinh^{-1}x \;\; then \; x = \sinh y$

Then $x = \frac{e^y - e^{-y}}{2}$

$2x = e^y - e^{-y}$

$2xe^y = e^{2y} - 1$

$0 = e^{2y} - 2xe^y - 1$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$e^y = x \pm \sqrt{x^2 + 1}$ but $e^y > 0 \; so \; take \; the \; positive \; root.$

$e^y = x + \sqrt{x^2 + 1}$

$y = \ln\left(x + \sqrt{x^2 + 1}\right)$

i.e. $sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$

Similarly we can show:-

$cosh^{-1}x = \ln(x + \sqrt{x^2 - 1})$ for $x \geq 1$

$tanh^{-1}x = \frac{1}{2} ln\left(\frac{1+x}{1-x}\right)$ for $|x| < 1$

These results are given in the formula booklet

Example Express (a.) $\operatorname{arcsinh}\frac{3}{4}$ (b.) $\operatorname{arccosh} 3$ (c.) $\operatorname{arctanh}\frac{-3}{4}$ in log form.

Example

Solve $sinh^2x + 5 = 4\cosh x$

**P3 book Ex4A Q26,27,29,31,32,33,35,38,40

The Derivatives of Hyperbolic Functions

$$\frac{d(\sinh x)}{dx} = \frac{d(\frac{e^x - e^{-x}}{2})}{dx}$$

$$\therefore \frac{d(\sinh x)}{dx} = \frac{1}{2}(e^x + e^{-x})$$

$$\therefore \frac{d(\sinh x)}{dx} = \cosh x$$

Also

$$\frac{d(\cosh x)}{dx} = \frac{d(\frac{e^x + e^{-x}}{2})}{dx}$$

$$\therefore \frac{d(\cosh x)}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$\therefore \frac{d(\cosh x)}{dx} = \sinh x$$

Example

Find $\frac{d(\tanh x)}{dx}$

Example

Find $\frac{d(\coth x)}{dx}$

Example

Find $\frac{d(\operatorname{sech} x)}{dx}$

Example

Find $\frac{d(\operatorname{cosech} x)}{dx}$

Example

Given $y = \cos x \cosh x, \ \ find \ \frac{d^2y}{dx^2}$.

Example

A curve is given by the equations $x = \cosh t$, $y = \sinh t$ where t is a parameter.

(a.) Find the cartesian equation of the curve.

(b.) Find the equation of the tangent at point where $t = \ln 2$.

The Derivatives of Inverse Hyperbolic Functions

1. $y = \sinh^{-1} x$

$$x = \sinh y$$

$$\frac{dx}{dy} = \cosh y$$
$$\frac{dy}{dx} = \frac{1}{\cosh y}$$
$$\frac{dy}{dx} = \frac{1}{\sqrt{\sinh^2 y + 1}}$$

*take the positive sign as $\cosh y$ is positive for all y and use $'\cosh^2 y - \sinh^2 y = 1'$ to get…

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}$$

$$\therefore \frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2+1}}$$

2. $y = \cosh^{-1} x$

$$x = \cosh y$$

$$\frac{dx}{dy} = \sinh y$$
$$\frac{dy}{dx} = \frac{1}{\sinh y}$$
$$\frac{dy}{dx} = \frac{1}{\pm\sqrt{\cosh^2 y + 1}}$$
$$\frac{dy}{dx} = \frac{1}{\pm\sqrt{x^2-1}}$$

(but $\cosh^{-1} x$ is defined for $y \geq 0$ so $\sinh y \geq 0$)

$$\therefore \frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2-1}}$$

3. $y = \tanh^{-1} x$

$$x = \tanh y$$

$$\frac{dx}{dy} = \operatorname{sech}^2 y$$
$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$$

Remember $1 - \text{sech}^2 y = \tanh^2 y \quad so \quad \frac{dy}{dx} = \frac{1}{1-\tanh^2 y}$

$$\frac{dy}{dx} = \frac{1}{1 - x^2}$$

$$\therefore \frac{d(\tanh^{-1} x)}{dx} = \frac{1}{1 - x^2}$$

$$4. \frac{d\left(\sinh^{-1}\left(\frac{x}{a}\right)\right)}{dx} =$$

$$5. \frac{d\left(\cosh^{-1}\left(\frac{x}{a}\right)\right)}{dx} =$$

$$6. \frac{d\left(\tanh^{-1}\left(\frac{x}{a}\right)\right)}{dx} =$$

Results

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = sinh^{-1}\left(\frac{x}{a}\right) + c \quad or \quad \ln(x + \sqrt{x^2 + a^2})$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = cosh^{-1}\left(\frac{x}{a}\right) + c \quad or \quad \ln(x - \sqrt{x^2 + a^2}), \quad (x > a)$$

<u>Example</u> Find the equation of thhe tangent at the point where $x = \frac{-1}{2}$ to the curve with equation $\mathrm{y} = \tanh^{-1} x$.

*P3 book Ex4B Q1-19odds, 21-25,27-53odds,54,56-59

Finding the general equation of a first order differential equation in which the variables are separable.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\therefore \frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\therefore \int \frac{1}{g(y)}dy = \int f(x)dx + c$$

Example

Given that $y = 2$ at $x = 0$ and $\frac{dy}{dx} = y^2 + 4$, find y in terms of x.

Solution

*P3 book Ex8A Q18-22

First Order Linear Differential Equaations

A 1st order linear differential equation is of the form

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of *x* or constants.

An equation of this form is said to be exact when one side is the exact derivative of a product and the other side can be integrated wrt *x*.

If it is not exact then it can be made exact by multiplying through the equation by a function of *x.* This function is called the integrating factor.

Example

Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

Multiplying through by *x* gives…

$$x\frac{dy}{dx} + y = x^3$$

…making it exact.

$$\therefore \frac{d(xy)}{dx} = x^3$$

$$\therefore xy = \int x^3 dx$$

$$\therefore xy = \frac{x^4}{4} + c$$

In this case the integrating factor is *x*.

Note:- The integrating factor is given by $f(x)$ where $f(x) = e^{\int P dx}$.

i.e. in the last example $P = \frac{1}{x}$

$$\therefore f(x) = e^{\int \frac{1}{x} dx}$$

$$\therefore f(x) = e^{\ln x}$$

$$\therefore f(x) = x$$

So the linear equation $\frac{dy}{dx} + Py = Q$ can be solved by multiplying by the integgrating factor $e^{\int P dx}$, provided $e^{\int P dx}$ can be found and the function $Qe^{\int P dx}$ can be integrated wrt *x*.

<u>Example</u>
Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = \sin x\, cos^3\, x$$

<u>Solution</u>

Example

Find y in terms of x given that

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \ln x \ \ for\ x > 0\ and\ y = 2\ at\ x = 1$$

Solution

*P3 book Ex8C Q1-9,13,14,16,17,18

The Second Order Linear Differential Equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0\ where\ a, b\ and\ c\ are\ constants$$

The equation is called the 2nd order, because its highest derivative of y wrt x is $\frac{d^2y}{dx^2}$.

The equation is called linear because only 1st degree terms in y and its derivatives occur.

Result: The general solution of the 2nd order differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ is $y=Au+Bv$,

where $y=u\ and\ y=v$ are particular, distinct solutions of the differential equation.

We now need to find the functions $u\ and\ v$ in specific cases.

In the differential equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, try as a solution…

$$y=e^{mx}\text{ where } m \text{ is a constant to be found.}$$

$$\frac{dy}{dx}=me^{mx}$$

$$\frac{d^2y}{dx^2}=m^2e^{mx}$$

If $y=e^{mx}$ is a solution of the differential equation then

$$am^2e^{mx}+b\,me^{mx}+ce^{mx}=0$$

$$\therefore am^2+b\,m+c=0\ (because\ e^{mx}>0\ for\ all\ m)$$

The 2 values of m required are the roots of the quadratic equation $am^2+b\,m+c=0$. This equation is called the Auxiliary Quadratic Equation and it may have..

(i) Real roots $(if\ b^2-4ac>0)$
(ii) Identical roots $(if\ b^2-4ac=0)$
(iii) Complex roots $(if\ b^2-4ac<0)$

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Solution

Generalising:- The general solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, whose auxiliary quadratic equation $am^2 + b\,m + c = 0$ has real distinct roots $\alpha\ and\ \beta$ is:-

$$y = Ae^{\alpha x} + Be^{\beta x}$$

(where A and B are constants)

*P3 Book Ex8D

Auxiliary Quadratic Equation With Real Coincident Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Solution

Generalising:- The general solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, whose auxiliary quadratic equation $am^2 + b\,m + c = 0$ has equal roots α is:-

$$y = (A + Bx)e^{\alpha x}$$

(where A and B are constants)

*P3 Book Ex8E

Auxiliary Quadratic Equation With Pure Imaginary Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

Solution

Result:- For the differential equation

$$\frac{d^2y}{dx^2} + n^2y = 0$$

General solution is $y = A\cos nx + B\sin nx$ (where A and B are constants).

Auxiliary Quadratic Equation With Complex Conjugate Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

Solution

Result:- For the differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where the auxiliary quadratic equation

$am^2 + b\,m + c = 0$ has complex conjugate roots

$p + iq \; and \; p - iq \; (where \; p \; and \; q \; \in R)$

the general solution is $y = e^{Px}(A\cos qx + B\sin qx)$

(where A and B are constants)

*P3 book Ex8F

The Second Order Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$

To solve this type of differential equation:-

Method:-

1. Solve the differential equation
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
The solution is called the *complementary function*.
2. Find a solution of the equation
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$$
$$where\ f(x) could\ be\ any\ one\ of\ these\ forms: -$$
   (i) A constant k
   (ii) A linear function $px + q$
   (iii) An exponential function ke^{px}
   (iv) A trig function e.g. $p\sin x, q\cos 2x\ or\ psin\ 3x + qcos3x$

A solution of the differential equation for any of the forms of $f(x)$ given above can be found by inspection.

This solution, when found, is called a *particular integral* of the equation.

3. The general solution of the diifferential equation is then
$$y = C.F. + P.I.$$

Examples on finding the P.I.

Example

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x)$$

Find P.I. of this differential equation in the cases where $f(x) = \cdots$

(a.) 12 (b.) $3x + 5$ (c.) $3e^{2x}$ (d.) $\cos 2x$

Solution

(a.)

(b.)

(c.)

(d.)

Example

Find y in terms of x for the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cos 2x$$

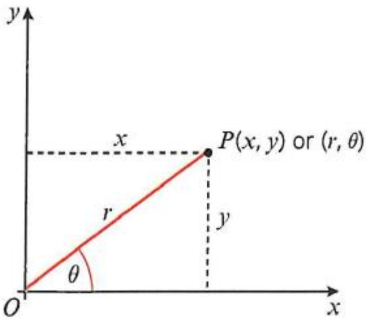
given that $\frac{dy}{dx} = 0\ at\ x = 0\ and\ y = 0\ at\ x = 0.$

Solution

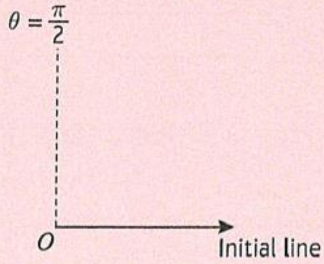
*P3 book Ex8G Q1-9odds,17,20,23,26-30

Polar Co-ordinates

Polar coordinates are an alternative way of describing the position of a point P in two-dimensional space. You need two measurements: firstly, the distance the point is from the **pole** (usually the origin O), r, and secondly, the angle measured anticlockwise from the **initial line** (usually the positive x-axis), θ. Polar coordinates are written as (r, θ).



Notation When working in polar coordinates the axes might also be labelled like this:



The coordinates of P can be written in either Cartesian form as (x, y) or in polar form as (r, θ).

You can convert between Cartesian coordinates and polar coordinates using right-angled triangle trigonometry.

From the diagram above you can see that:

- $r\cos\theta = x$
  $r\sin\theta = y$
- $r^2 = x^2 + y^2$
  $\theta = \arctan\left(\frac{y}{x}\right)$

Watch out Always draw a sketch diagram to check in which quadrant the point lies, and always measure the polar angle from the positive x-axis.

Example

Find polar coordinates of the points with the following Cartesian coordinates.

a $(3, 4)$ **b** $(5, -12)$ **c** $(-\sqrt{3}, -1)$

Example

Convert the following polar coordinates into Cartesian form. The angles are measured in radians.

a $\left(10, \frac{4\pi}{3}\right)$ **b** $\left(8, \frac{2\pi}{3}\right)$

Polar equations of curves are usually given in the form $r = \mathrm{f}(\theta)$. For example,

$r = 2\cos\theta$

$r = 1 + 2\theta$

$r = 3$ — In this example r is constant.

You can convert between polar equations of curves and their Cartesian forms.

Example

Find Cartesian equations of the following curves.

a $r = 5$ **b** $r = 2 + \cos 2\theta$ **c** $r^2 = \sin 2\theta, \quad 0 < \theta \leqslant \frac{\pi}{2}$

Example

Find polar equations for the following:

a $y^2 = 4x$ **b** $x^2 - y^2 = 5$ **c** $y\sqrt{3} = x + 4$

Sketching Curves

You can sketch curves given in polar form by learning the shapes of some standard curves.

- **$r = a$ is a circle with centre O and radius a.**
- **$\theta = \alpha$ is a half-line through O and making an angle α with the initial line.**
- **$r = a\theta$ is a spiral starting at O.**

Example

Sketch the following curves.

a $r = 5$ **b** $\theta = \frac{3\pi}{4}$ **c** $r = a\theta$

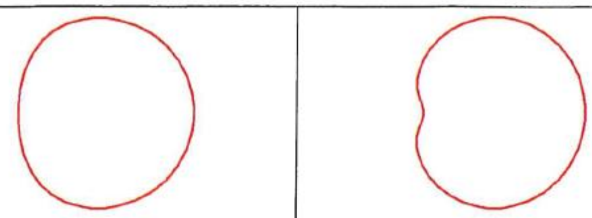
where a is a positive constant.

Example

Sketch the following curves.

a $r = a(1 + \cos\theta)$ **b** $r = a\sin 3\theta$ **c** $r^2 = a^2\cos 2\theta$

Curves with equations of the form $r = a(p + q\cos\theta)$ are defined for all values of θ if $p \geqslant q$. An example of this, when $p = q$, was the cardioid seen in Example 6a. These curves fall into two types, those that are 'egg' shaped (i.e. a convex curve) and those with a 'dimple' (i.e. the curve is concave at $\theta = \pi$). The conditions for each type are given below:



'egg' shape when $p \geqslant 2q$	'dimple' shape when $q \leqslant p < 2q$

Links You can prove these conditions by considering the number of tangents to the curve that are perpendicular to the initial line.
→ Example 14

Example

Sketch the following curves.

a $r = a(5 + 2\cos\theta)$ **b** $r = a(3 + 2\cos\theta)$

You may also need to find a polar curve to represent a locus of points on an Argand diagram.

Links If the pole is taken as the origin, and the initial line is taken as the positive real axis, then the point (r, θ) will represent the complex number $re^{i\theta}$ ← Section 1.1

Example

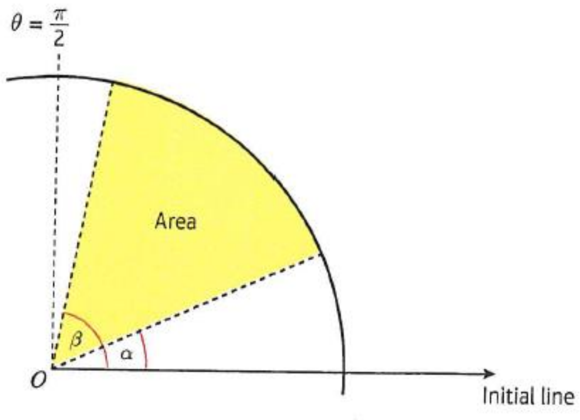
a Show on an Argand diagram the locus of points given by the values of z satisfying
$|z - 3 - 4i| = 5$

b Show that this locus of points can be represented by the polar curve $r = 6\cos\theta + 8\sin\theta$.

Area Enclosed By A Curve

You can find areas enclosed by a polar curve using integration.

- **The area of a sector bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by the formula**
  Area $= \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$



Example

Find the area enclosed by the cardioid with equation $r = a(1 + \cos\theta)$.

Example

Find the area of one loop of the curve with polar equation $r = a\sin 4\theta$.

Watch out $r = \sin n\theta$ has n loops and so a simple way of finding the area of one loop would appear to be to find $\frac{1}{2}\int_0^{2\pi} r^2\,d\theta$ and divide by n. This would give $\frac{a^2\pi}{8}$

The reason why this is not the correct answer is because when you take r^2 in the integral you are also including the n loops given by $r < 0$. You need to choose your limits carefully so that $r \geqslant 0$ for all values within the range of the integral.

Example

a On the same diagram, sketch the curves with equations $r = 2 + \cos\theta$ and $r = 5\cos\theta$.

b Find the polar coordinates of the points of intersection of these two curves.

c Find the exact area of the region which lies within both curves.

Tangents To Polar Curves

If you are given a curve $r = \mathrm{f}(\theta)$ in polar form, you can write it as a parametric curve in Cartesian form, using θ as the parameter:

$$x = r\cos\theta = \mathrm{f}(\theta)\cos\theta$$

$$y = r\sin\theta = \mathrm{f}(\theta)\sin\theta$$

By differentiating parametrically, you can find the gradient of the curve at any point:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}$$

When $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$, a tangent to the curve will be horizontal.

When $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$, a tangent to the curve will be vertical.

You need to be able to find tangents to a polar curve that are **parallel** or **perpendicular** to the initial line.

- **To find a tangent parallel to the initial line set $\frac{\mathrm{d}y}{\mathrm{d}\theta} = 0$.**
- **To find a tangent perpendicular to the initial line set $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 0$.**

Example

Find the coordinates of the points on $r = a(1 + \cos\theta)$ where the tangents are parallel to the initial line $\theta = 0$.

Example

Find the equations and the points of contact of the tangents to the curve $r = a \sin 2\theta$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$ that are:

a parallel to the initial line

b perpendicular to the initial line.

Give answers to three significant figures where appropriate.

Summary of key points

1 For a point P with polar coordinates (r, θ) and Cartesian coordinates (x, y),

- $r\cos\theta = x$ and $r\sin\theta = y$
- $r^2 = x^2 + y^2$, $\theta = \arctan\left(\frac{y}{x}\right)$

Care must be taken to ensure that θ is in the correct quadrant.

2
- $r = a$ is a circle with centre O and radius a.
- $\theta = \alpha$ is a half-line through O and making an angle α with the initial line.
- $r = a\theta$ is a spiral starting at O.

3 The **area of a sector** bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by the formula

$$\textbf{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

4
- To find a tangent parallel to the initial line set $\frac{dy}{d\theta} = 0$.
- To find a tangent perpendicular to the initial line set $\frac{dx}{d\theta} = 0$.