

The exponential functions can be combined to form functions that have strong similarities to trig (or circular) functions. These functions are called hyperbolic cosine (cosh x) and hyperbolic sine (sinh x).

$$\cosh x = \frac{e^x + e^{-x}}{2} \text{ for } x \in R \quad \text{similar to } \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ for } x \in R \quad \text{similar to } \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

These two definitions are basic and from them four other hyperbolic functions are defined:-

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\therefore \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \quad \text{for } x \in R$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \quad \text{for } x \in R$$

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} \quad \text{for } x \in R, x \neq 0$$

$$\operatorname{coth} x = \frac{1}{\tanh x} = \frac{e^{2x} + 1}{e^{2x} - 1} \quad \text{for } x \in R, x \neq 0$$

Graphs of Hyperbolic Functions

$$\sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^x}{2} = \frac{-(e^x - e^{-x})}{2} = -\sinh x$$

So $\sinh x$ is an odd function.

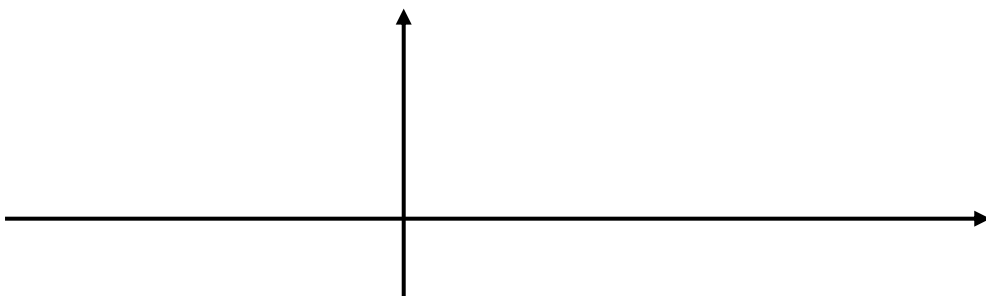
Similarly

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

So $\cosh x$ is an even function.

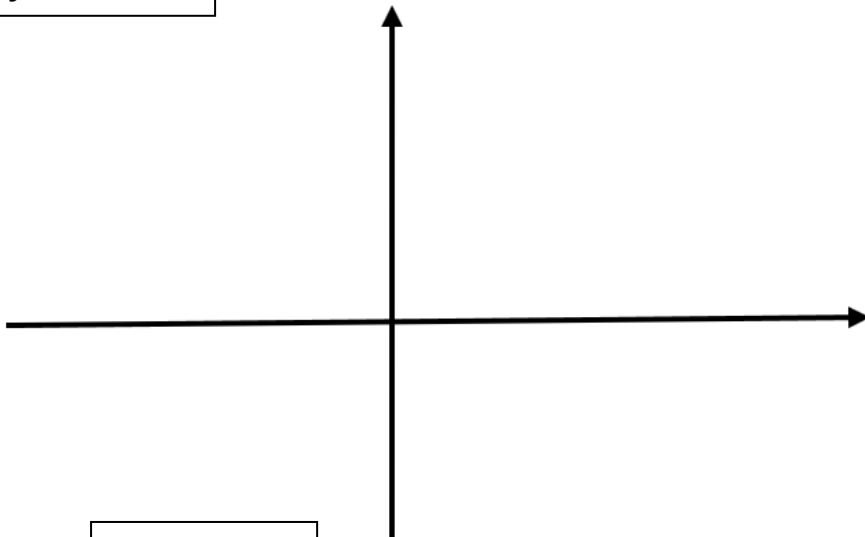
Also $\cosh x = \frac{e^x + e^{-x}}{2} > \frac{e^x - e^{-x}}{2} = \sinh x$ for all values.

First sketch $y = e^x$ and $y = e^{-x}$



So

$y = \cosh x$



$y = \sinh x$

Since $\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$, we see at $x = 0$, $\tanh x = 0$.

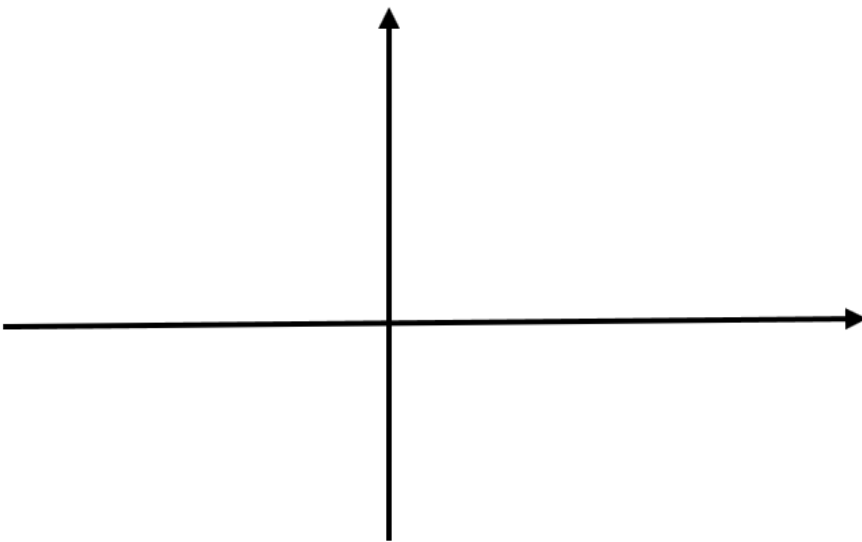
$$\text{Also, } \tanh(-x) = \frac{e^{-2x}-1}{e^{-2x}+1} = \frac{\frac{1}{e^{2x}}-1}{\frac{1}{e^{2x}}+1} = \frac{1-e^{2x}}{1+e^{2x}} = -\tanh x$$

So $\tanh x$ is an odd function.

$$\text{Now } \tanh x = \frac{e^{2x}-1}{e^{2x}+1} = \frac{1-e^{-2x}}{1+e^{-2x}} \quad (\text{by dividing through by } e^{2x})$$

As $x \rightarrow \infty$, $e^{-2x} \rightarrow 0$ and $\tanh x \rightarrow 1$

As $x \rightarrow -\infty$, $e^{2x} \rightarrow 0$ and $\tanh x \rightarrow -1$



The lines $y = \pm 1$ are asymptotes to the curve.

Example Sketch $y = \operatorname{sech} x$ for $x \in \mathbb{R}$.



Example Find the exact values of x for which $\tanh x = \frac{1}{2}$.

Identities

Example Prove $\cosh^2 x - \sinh^2 x \equiv 1$

Example Prove $\cosh(x + y) \equiv \cosh x \cosh y + \sinh x \sinh y$

Example Find an identity for $\sinh 2A$ in terms of $\cosh A$ and $\sinh A$. Hence find an identity for $\tanh 2A$.

Osborne's Rule:- The formulae for circular and hyperbolic functions correspond exactly, provided the sign is changed whenever there exists a product (or implied product) of 2 sines.

i.e. the rule is to replace each trig function with its corresponding hyperbolic function and change the sign of every product (or implied product) of 2 sines.

$$\text{e.g. } \cos 2A = 1 - \sin^2 A$$

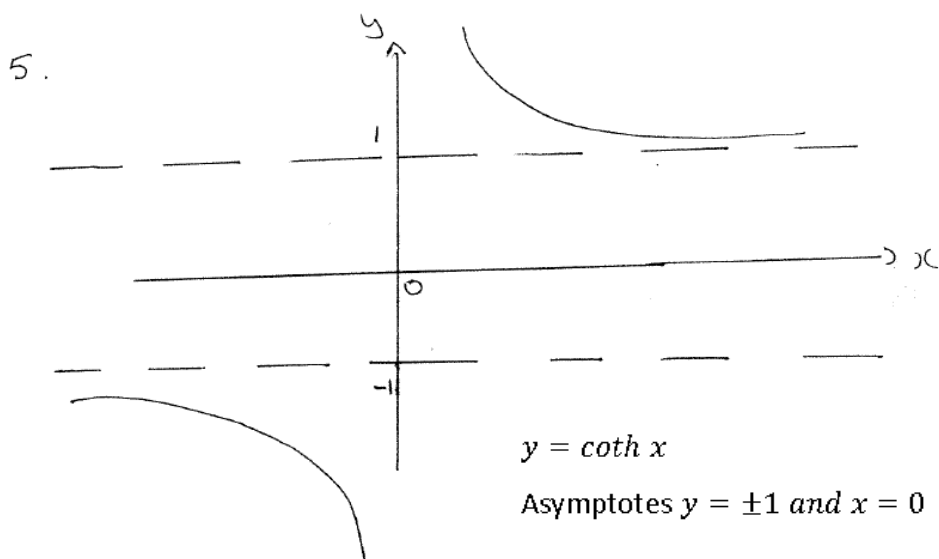
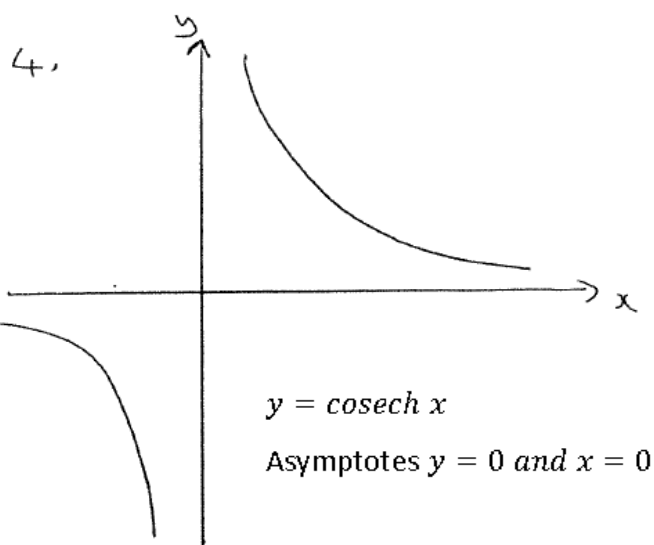
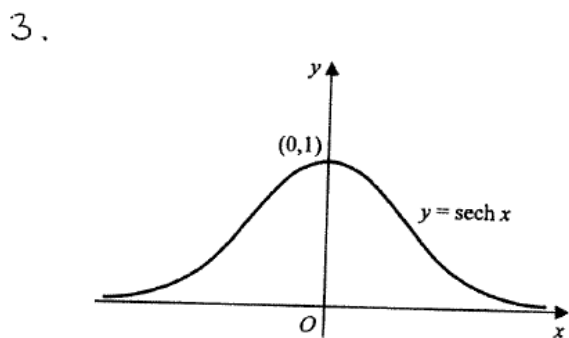
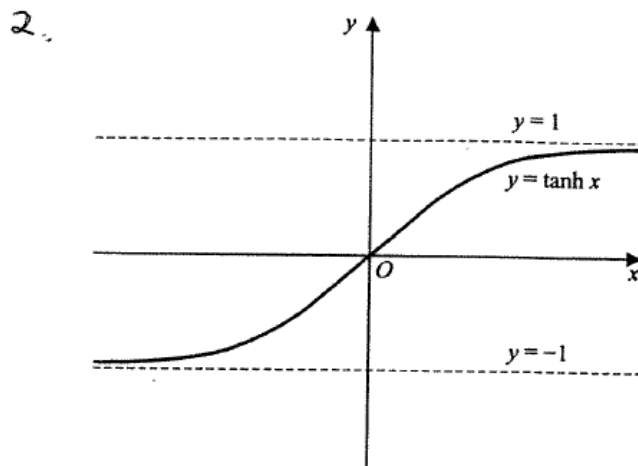
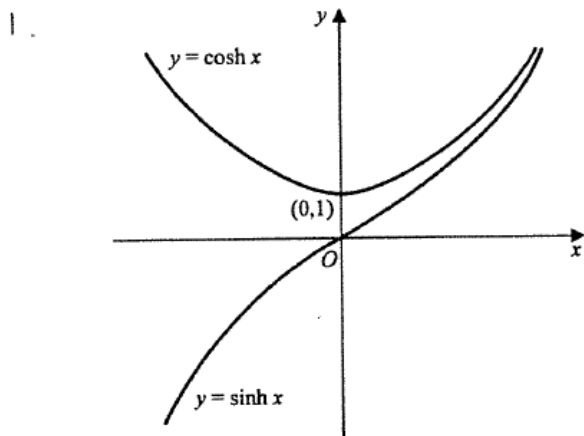
$$\text{becomes } \cosh 2A = 1 + \sinh^2 A$$

$$\text{e.g. } \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\text{becomes } \tanh(A - B) = \frac{\tanh A - \tanh B}{1 - \tanh A \tanh B}$$

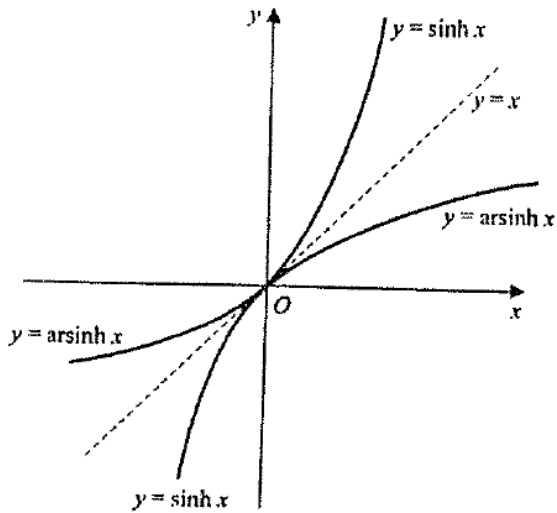
**P3 book Ex4A Q(1,2,3)alt parts, 4,5,7-17odds,18,20,22,23,25

Graphs of Hyperbolic Functions

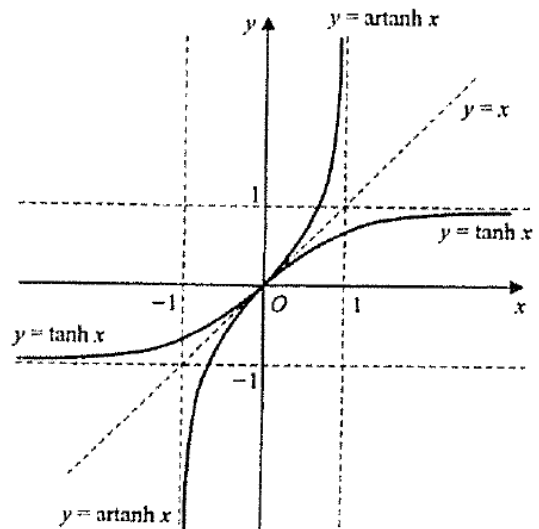


Inverse Hyperbolic Functions

1.

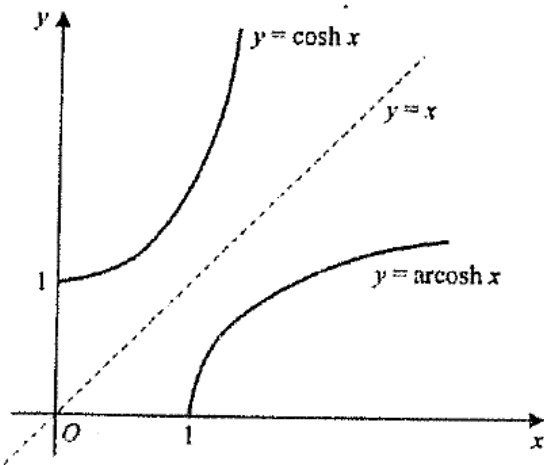


2.



3.

For the function $\cosh x$, you need to take the domain $x \geq 0$, so that it is a one-one function. Then the inverse function $\operatorname{arcosh} x$ is defined for the domain $x \geq 1$ and range $\operatorname{arcosh} x \geq 0$. The graphs of $\cosh x$ and $\operatorname{arcosh} x$ look like this:



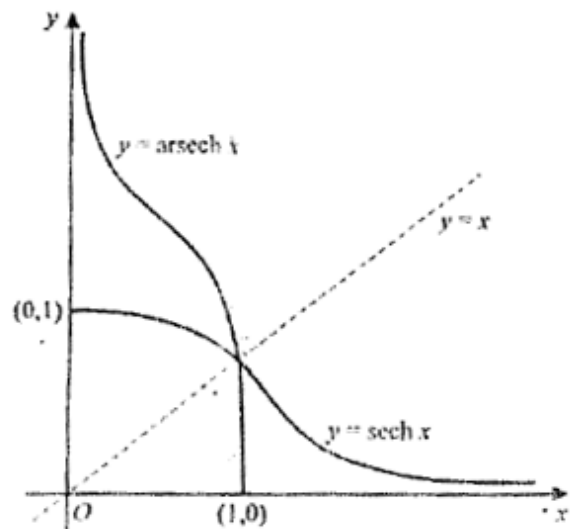
4.

In the same diagram, sketch the curves

$$y = \operatorname{sech} x, x \in \mathbb{R}, x \geq 0$$

$$y = \operatorname{arsech} x, x \in \mathbb{R}, 0 < x \leq 1$$

The curves are shown in the diagram. One is the reflection of the other in the line $y = x$.



The Logarithmic Form of Inverse Hyperbolic Functions

If $y = \sinh^{-1}x$ then $x = \sinh y$

$$\text{Then } x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2xe^y = e^{2y} - 1$$

$$0 = e^{2y} - 2xe^y - 1$$

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2}$$

$e^y = x \pm \sqrt{x^2 + 1}$ but $e^y > 0$ so take the positive root.

$$e^y = x + \sqrt{x^2 + 1}$$

$$y = \ln(x + \sqrt{x^2 + 1})$$

i.e. $\sinh^{-1}x = \ln(x + \sqrt{x^2 + 1})$

Similarly we can show:-

$$\cosh^{-1}x = \ln(x + \sqrt{x^2 - 1}) \text{ for } x \geq 1$$

$$\tanh^{-1}x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \text{ for } |x| < 1$$

****These results are given in the formula booklet****

Example Express (a.) $\operatorname{arcsinh} \frac{3}{4}$ (b.) $\operatorname{arccosh} 3$ (c.) $\operatorname{arctanh} \frac{-3}{4}$ in log form.

Example

Solve $\sinh^2 x + 5 = 4 \cosh x$

**P3 book Ex4A Q26,27,29,31,32,33,35,38,40

The Derivatives of Hyperbolic Functions

$$\frac{d(\sinh x)}{dx} = \frac{d\left(\frac{e^x - e^{-x}}{2}\right)}{dx}$$

$$\therefore \frac{d(\sinh x)}{dx} = \frac{1}{2}(e^x + e^{-x})$$

$$\therefore \frac{d(\sinh x)}{dx} = \cosh x$$

Also

$$\frac{d(\cosh x)}{dx} = \frac{d\left(\frac{e^x + e^{-x}}{2}\right)}{dx}$$

$$\therefore \frac{d(\cosh x)}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$\therefore \frac{d(\cosh x)}{dx} = \sinh x$$

Example

Find $\frac{d(\tanh x)}{dx}$

Example

Find $\frac{d(\coth x)}{dx}$

Example

Find $\frac{d(\operatorname{sech} x)}{dx}$

Example

Find $\frac{d(\operatorname{cosech} x)}{dx}$

Example

Given $y = \cos x \cosh x$, find $\frac{d^2y}{dx^2}$.

Example

A curve is given by the equations $x = \cosh t$, $y = \sinh t$ where t is a parameter.

- (a.) Find the cartesian equation of the curve.
- (b.) Find the equation of the tangent at point where $t = \ln 2$.

The Derivatives of Inverse Hyperbolic Functions

1. $y = \sinh^{-1} x$

$$x = \sinh y$$

$$\frac{dx}{dy} = \cosh y$$

$$\frac{dy}{dx} = \frac{1}{\cosh y}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\sinh^2 y + 1}}$$

*take the positive sign as $\cosh y$ is positive for all y and use ' $\cosh^2 y - \sinh^2 y = 1$ ' to get...

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

$$\therefore \frac{d(\sinh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2 + 1}}$$

2. $y = \cosh^{-1} x$

$$x = \cosh y$$

$$\frac{dx}{dy} = \sinh y$$

$$\frac{dy}{dx} = \frac{1}{\sinh y}$$

$$\frac{dy}{dx} = \frac{1}{\pm \sqrt{\cosh^2 y + 1}}$$

$$\frac{dy}{dx} = \frac{1}{\pm \sqrt{x^2 - 1}}$$

(but $\cosh^{-1} x$ is defined for $y \geq 0$ so $\sinh y \geq 0$)

$$\therefore \frac{d(\cosh^{-1} x)}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

3. $y = \tanh^{-1} x$

$$x = \tanh y$$

$$\frac{dx}{dy} = \operatorname{sech}^2 y$$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$$

Remember $1 - \operatorname{sech}^2 y = \tanh^2 y$ so $\frac{dy}{dx} = \frac{1}{1 - \tanh^2 y}$
 $\frac{dy}{dx} = \frac{1}{1 - x^2}$

$$\therefore \frac{d(\tanh^{-1} x)}{dx} = \frac{1}{1 - x^2}$$

$$4. \frac{d\left(\sinh^{-1}\left(\frac{x}{a}\right)\right)}{dx} =$$

$$5. \frac{d\left(\cosh^{-1}\left(\frac{x}{a}\right)\right)}{dx} =$$

$$6. \frac{d\left(\tanh^{-1}\left(\frac{x}{a}\right)\right)}{dx} =$$

Results

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \sinh^{-1} \left(\frac{x}{a} \right) + c \quad \text{or} \quad \ln(x + \sqrt{x^2 + a^2})$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1} \left(\frac{x}{a} \right) + c \quad \text{or} \quad \ln(x - \sqrt{x^2 + a^2}), \quad (x > a)$$

Example Find the equation of the tangent at the point where $x = \frac{-1}{2}$ to the curve with equation $y = \tanh^{-1} x$.