A21 Further Maths

Differentiation and Integration of Inverse Trig Functions

Graphs of inverse trigonometric functions

$y = \arcsin x$

Remember we defined

$$y = \sin x$$

to have domain

$$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$$

and the range is

$$-1 \leqslant \sin x \leqslant 1$$

Our inverse

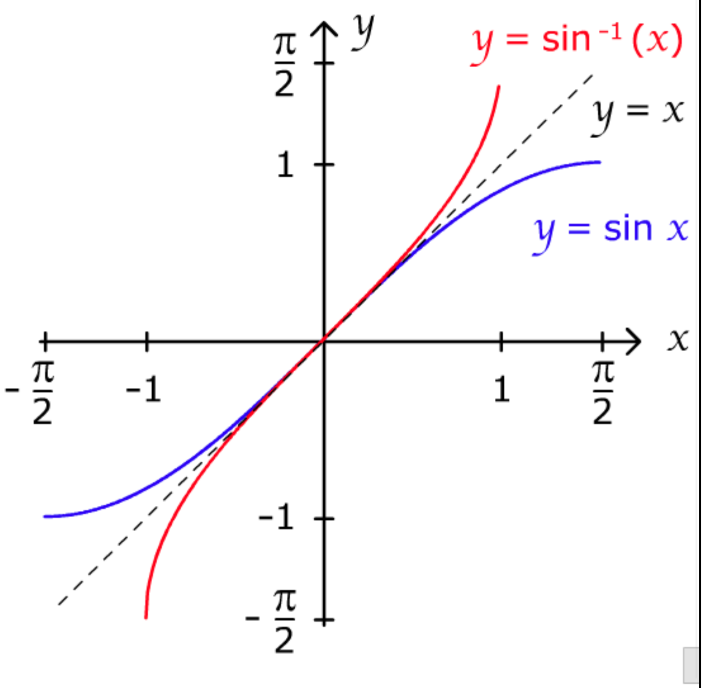
$$y = \sin^{-1} x$$

has domain

$$-1 \leqslant x \leqslant 1$$

and range

$$-\frac{\pi}{2} \leqslant \sin^{-1} x \leqslant \frac{\pi}{2}$$



$y = \arccos x$

We limit the domain to

$$0 \leqslant x \leqslant \pi$$

and the range is

$$-1 \leqslant \cos x \leqslant 1$$

The inverse function looks like this.

It's a reflection of $y = \cos x$ in the line $y = x$.

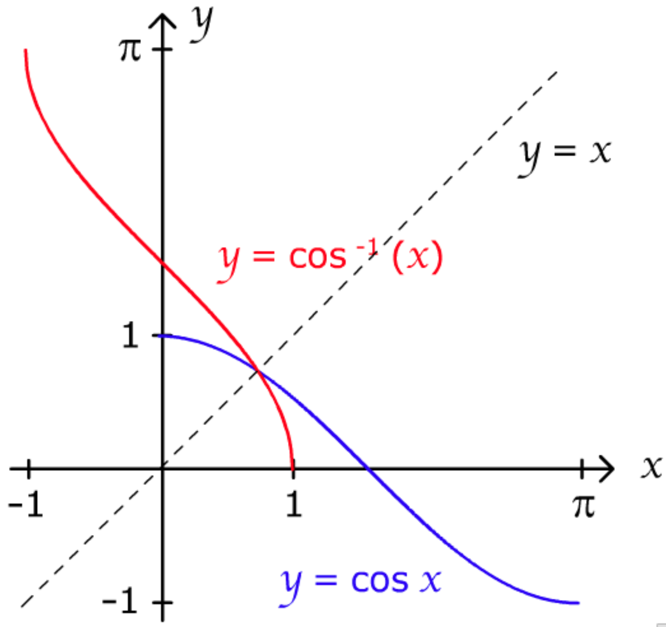
The domain of

$$y = \cos^{-1}(x)$$

is $\quad -1 \leqslant x \leqslant 1$

and the range is

$$0 \leqslant \cos^{-1}(x) \leqslant \pi$$



$y = \arctan x$

Limit it to

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

Reflect it in the line $y = x$ ...

... to get the inverse $y = \tan^{-1}(x)$.

The asymptotes are also reflected.

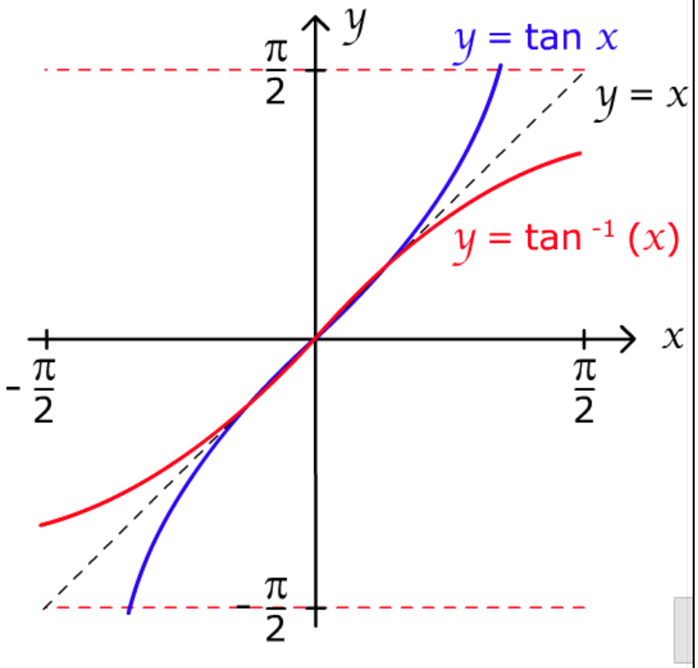
The domain of

$$y = \tan^{-1}(x)$$

is $x \in \mathbb{R}$

and the range is

$$-\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$$



Differentiation of Inverse Trig Functions

1. Let $y = \sin^{-1} x \;\therefore\; x = \sin y \qquad -1 \leq x \leq 1 \text{ and } \frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

$$\therefore \cos y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\pm\sqrt{1 - x^2}}$$

$* \; But \; y = \sin^{-1} x \; is \; an \; increasing \; function \; between$

$$-1 \; and \; 1, so \; \frac{dy}{dx} \; is \; positive.$$

$$\therefore \frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

2. Let $y = \cos^{-1} x \ \therefore x = \cos y \qquad -1 \le x \le 1$ and $0 \le y \le \pi$

$\therefore \ -\sin y \dfrac{dy}{dx} = 1$

$\therefore \dfrac{dy}{dx} = \dfrac{-1}{\sin y}$

$\therefore \dfrac{dy}{dx} = \dfrac{-1}{\pm\sqrt{1-x^2}}$

$*$ *But* $y = \cos^{-1} x$ *is a decreasing function between* -1 *and* 1, *so* $\dfrac{dy}{dx}$ *is negative.*

$\therefore \dfrac{d(\cos^{-1} x)}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$

3. Let $y = \tan^{-1} x \ \therefore x = \tan y \ -\infty < x < \infty$ (or $x \in R$)and $\frac{-\pi}{2} < y < \frac{\pi}{2}$

$\therefore \ \sec^2 y \dfrac{dy}{dx} = 1$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{\sec^2 y}$

$* \sec^2 y = 1 + \tan^2 y$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{1+x^2}$

$\therefore \dfrac{d(\tan^{-1} x)}{dx} = \dfrac{1}{1+x^2}$

Results

$$\frac{d(sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\cos^{-1} x)}{dx} = \frac{-1}{\sqrt{1-x^2}},$$

$$\frac{d(tan^{-1} x)}{dx} = \frac{1}{1+x^2}$$

Example Find $\frac{dy}{dx}$ when

(a.) $y = \cos^{-1} x^2$

(b.) $y = \tan^{-1}(e^{3x})$

Example Find an equation of the normal to the curve $y = \sin^{-1} 2x$ at point where $x = \frac{1}{4}$.

*P3 Book Ex3C Q7-15

Integration of $\frac{1}{a^2+x^2}$ ***and*** $\frac{1}{\sqrt{a^2-x^2}}$

1. Since $\frac{d(\sin^{-1}x)}{dx} = \frac{1}{\sqrt{1-x^2}}$

then $\frac{d(\sin^{-1}\frac{x}{a})}{dx} = \frac{1\times\frac{1}{a}}{\sqrt{1-\left(\frac{x}{a}\right)^2}}$

$\therefore \frac{d(\sin^{-1}\frac{x}{a})}{dx} = \frac{1}{a\sqrt{1-\left(\frac{x}{a}\right)^2}}$ $\qquad \therefore \frac{d(\sin^{-1}\frac{x}{a})}{dx} = \frac{1}{\sqrt{a^2-x^2}}$

$hence \int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a} + c$

2. And since $\frac{d(\tan^{-1}x)}{dx} = \frac{1}{1+x^2}$

then $\frac{d(\tan^{-1}\frac{x}{a})}{dx} = \frac{1\times\frac{1}{a}}{1+\left(\frac{x}{a}\right)^2}$

$\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx} = \frac{1}{a(1+\left(\frac{x}{a}\right)^2)}$

$\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx} = \frac{1}{a\left(\frac{a^2+x^2}{a^2}\right)}$

$\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx} = \frac{1}{a\left(\frac{a^2+x^2}{a^2}\right)}$

$\therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx} = \frac{1}{\left(\frac{a^2+x^2}{a}\right)}$ $\qquad \therefore \frac{d(\tan^{-1}\frac{x}{a})}{dx} = \frac{a}{a^2+x^2}$

$hence \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

Example Find (a.) $\int \frac{4}{x^2+16} dx$ (b.) $\int \frac{2}{36+x^2} dx$

Example Evaluate $\int_{-1.5}^{0} \frac{1}{\sqrt{9-x^2}} dx$

*UPM Ex15H Q13-24