A-Level Further Maths A21

## Maclaurins Series

Let f(x) be a function, which throughout a certain domain, including x=0 is

(a.) Differentiable any number of times ,and
(b.) The sum of a convergent power series.

Let this series be

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$so\ f(0) = a_0$$

*differentiating term by term and putting x=0

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$$

$$so\ f'(0) = a_1$$

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots$$

$$so\ f''(0) = 2a_2$$

$$or\ f''(0) = 2!\,a_2$$

$$f'''(x) = 6a_3 + 24a_4x + 60a_5x^2 + \cdots$$

$$so\ f'''(0) = 6a_3$$

$$or\ f'''(0) = 3!\,a_2$$

$$so\ you\ could\ write\ f(x) = f(0) + xf'(0) + \frac{x^2f''(0)}{2!} + \frac{x^3f'''(0)}{3!} + \frac{x^4f'^{v}(0)}{4!} + \cdots + \frac{x^nf^n(0)}{n!} + \cdots$$

This is Maclaurins Series.

Exponential Series

$Let\ f(x) = e^x$

Logarithmic Series

$Let\ f(x) = \ln(1 + x)$

Example

Expand $\cos x$ in ascending powers of x.

Solution

***P3 Book Exercise 2D

(next bit is not needed. Just to show)

**Challenge**

The **ratio test** is a sufficient condition for the convergence of an infinite series. It says that a series $\sum_{r=1}^{\infty} a_r$ converges if $\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| < 1$, and diverges if $\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| > 1$.

Use the ratio test to show that

**a** the Maclaurin series expansion of $e^x$ converges for all $x \in \mathbb{R}$

**Problem-solving**

If $\lim_{r\to\infty}\left|\frac{a_{r+1}}{a_r}\right| = 1$ or does not exist then the ratio test is inconclusive.

$$e^x = \sum_{r=1}^{\infty} \frac{x^r}{r!}$$

$$= \lim_{r\to\infty}\left|\frac{\frac{x^{r+1}}{(r+1)!}}{\frac{x^r}{r!}}\right| = \lim_{r\to\infty}\left|\frac{x}{(r+1)}\right|$$ which is <1 for all x.

# Binomial Series

Consider $f(x) = (1+x)^n \quad for\, n \in R$

$$f(x) = (1+x)^n \quad so\, f(0) = 1$$

$$f'(x) = n(1+x)^{n-1} \quad so\, f'(0) = n$$

$$f''(x) = n(n-1)(1+x)^{n-2} \quad so\, f''(0) = n(n-1)$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \quad so\, f'''(0) = n(n-1)(n-2)$$

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$$f^r(x) = n(n-1)(n-2)..(n-r+1)(1+x)^r \quad so\, f^r(0) = n(n-1)(n-2)..(n-r+1)$$

Maclaurins gives:-

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + .. + \frac{n(n-1)(n-2)..(n-r+1)x^r}{r!}$$

Which is the Binomial Series for any $n \in R$ and is convergent, provided $|x| < 1$.

If $n \in Z^+$, the series terminates and reduces to the Binomial Theorem.

Note Define $\binom{n}{r}$ to be

$$\binom{n}{r} = \frac{n(n-1)(n-2)..(n-r+1)}{r!}$$

Example

Expand $(1-3x)^{\frac{-2}{3}}$ up to terms including $x^3$.

Solution

Example

Expand $(1-3x)^{\frac{1}{5}}$ inascending powers of x up to the term $x^3$. Take $x=\frac{1}{32}$ to find an approprimation for $29^{\frac{1}{5}}$, giving your answer correct to 5d.p.

Solution

Example

$$f(x)=\frac{x}{(3-2x)(2-x)}$$

(a.) Express $f(x)$ in partial fractions
(b.) Expand $f(x)$ up to terms including $x^3$.
(c.) State the set of values of x for which the series is valid.

Solution

**Exercise 2C Q17,20-25

Using the Polynomial Series Form of Functions To Find Approximations For The Functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

So if you take terms in $x^3$ and higher powers of x to be negligible, then

$\sin x \approx x$ and $\cos x \approx 1 - \frac{x^2}{2}$ where x is small.

Also

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

So for small x, $\tan x \approx x$.

Example

Find a quadratic polynomial approximation for $\frac{\sin 2x}{1+x}$, give that x is small.

Solution

Example

Given that x is small, show that $\frac{3\sin x}{2+\cos x} \approx x$.

Solution

Example

Show that $\lim_{x \to 0} \frac{1 - \cos 4x + x sin\ 3x}{x^2} = 11$

Solution

**pg 29 Exercise 2E (use the standard Maclaurins results in the formula booklet)