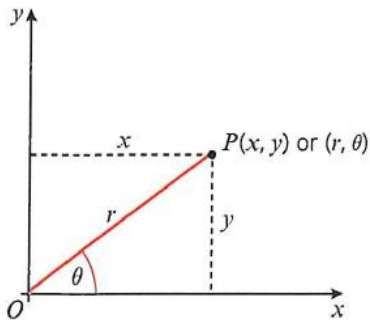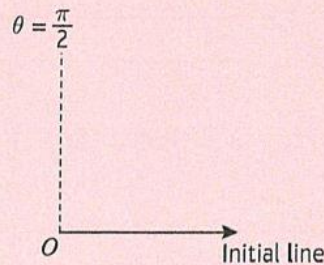# Polar Co-ordinates

Polar coordinates are an alternative way of describing the position of a point P in two-dimensional space. You need two measurements: firstly, the distance the point is from the **pole** (usually the origin O), r, and secondly, the angle measured anticlockwise from the **initial line** (usually the positive x-axis), θ. Polar coordinates are written as (r, θ).

Notation When working in polar coordinates the axes might also be labelled like this:

$\theta = \frac{\pi}{2}$

Initial line

The coordinates of P can be written in either Cartesian form as (x, y) or in polar form as (r, θ).

You can convert between Cartesian coordinates and polar coordinates using right-angled triangle trigonometry.

From the diagram above you can see that:

- $r\cos\theta = x$
  $r\sin\theta = y$

- $r^2 = x^2 + y^2$
  $\theta = \arctan\left(\dfrac{y}{x}\right)$

Watch out Always draw a sketch diagram to check in which quadrant the point lies, and always measure the polar angle from the positive x-axis.

Example

Find polar coordinates of the points with the following Cartesian coordinates.

a $(3, 4)$ 　　　　　　b $(5, -12)$ 　　　　　　c $(-\sqrt{3}, -1)$

Convert the following polar coordinates into Cartesian form. The angles are measured in radians.

a $\left(10, \frac{4\pi}{3}\right)$ **b** $\left(8, \frac{2\pi}{3}\right)$

Polar equations of curves are usually given in the form $r = f(\theta)$. For example,

$$r = 2\cos\theta$$
$$r = 1 + 2\theta$$
$$r = 3 \bullet\text{————————————————————} \text{In this example } r \text{ is constant.}$$

You can convert between polar equations of curves and their Cartesian forms.
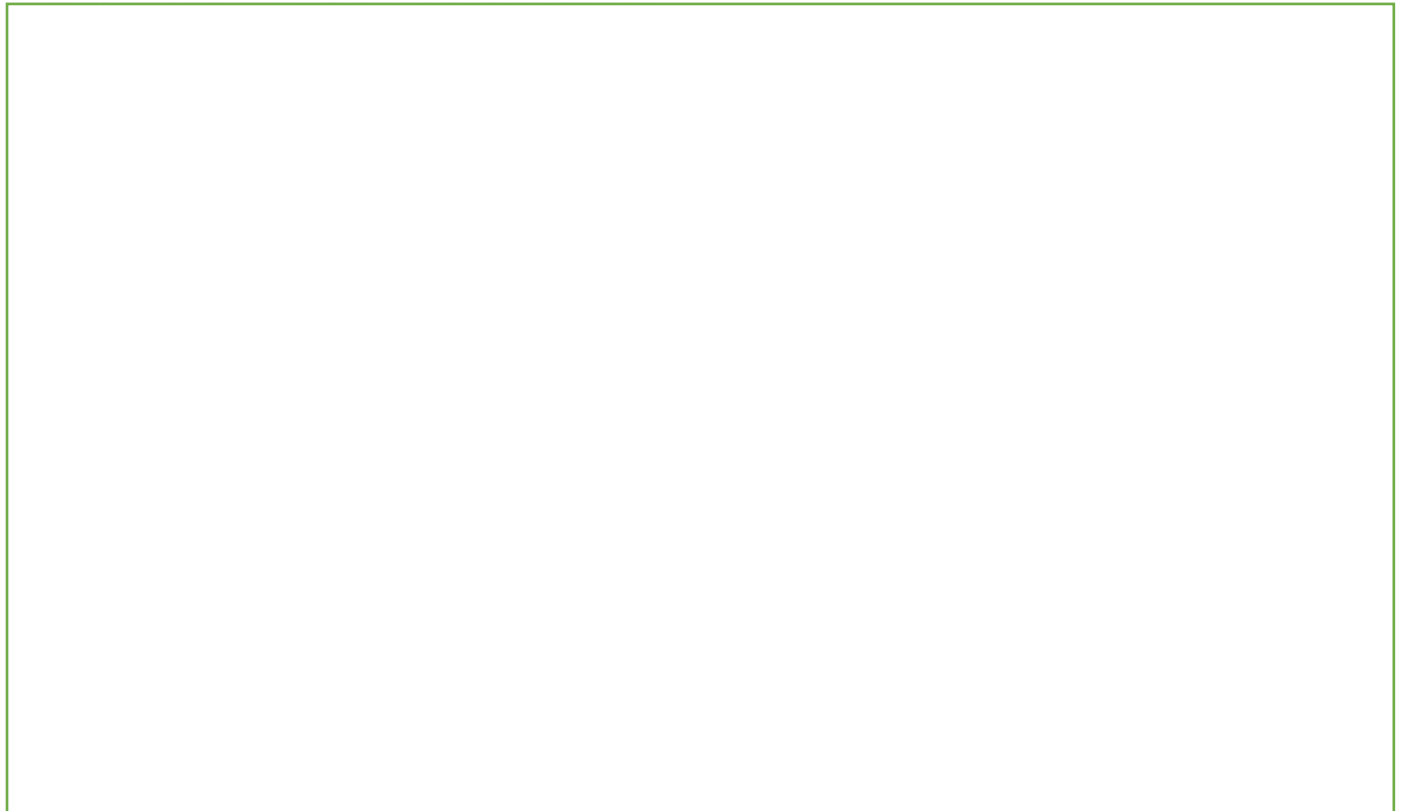
Find Cartesian equations of the following curves.

a $r = 5$ **b** $r = 2 + \cos 2\theta$ **c** $r^2 = \sin 2\theta, \quad 0 < \theta \leqslant \dfrac{\pi}{2}$
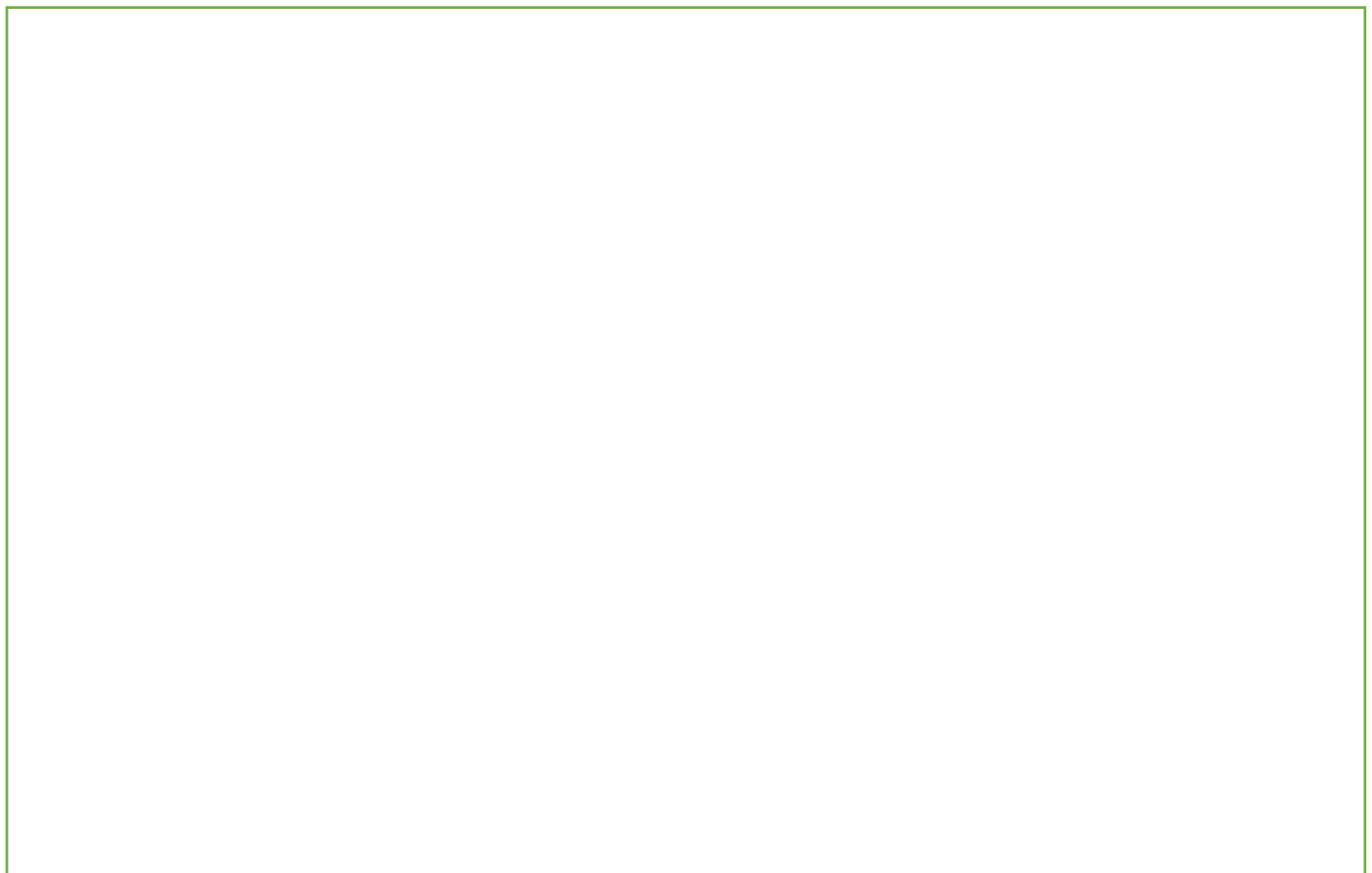
Find polar equations for the following:

a $y^2 = 4x$ **b** $x^2 - y^2 = 5$ **c** $y\sqrt{3} = x + 4$

Sketching Curves

You can sketch curves given in polar form by learning the shapes of some standard curves.

- $r = a$ **is a circle with centre** O **and radius** a.
- $\theta = \alpha$ **is a half-line through** O **and making an angle** α **with the initial line.**
- $r = a\theta$ **is a spiral starting at** O.

Example

Sketch the following curves.

a $r = 5$ **b** $\theta = \dfrac{3\pi}{4}$ **c** $r = a\theta$
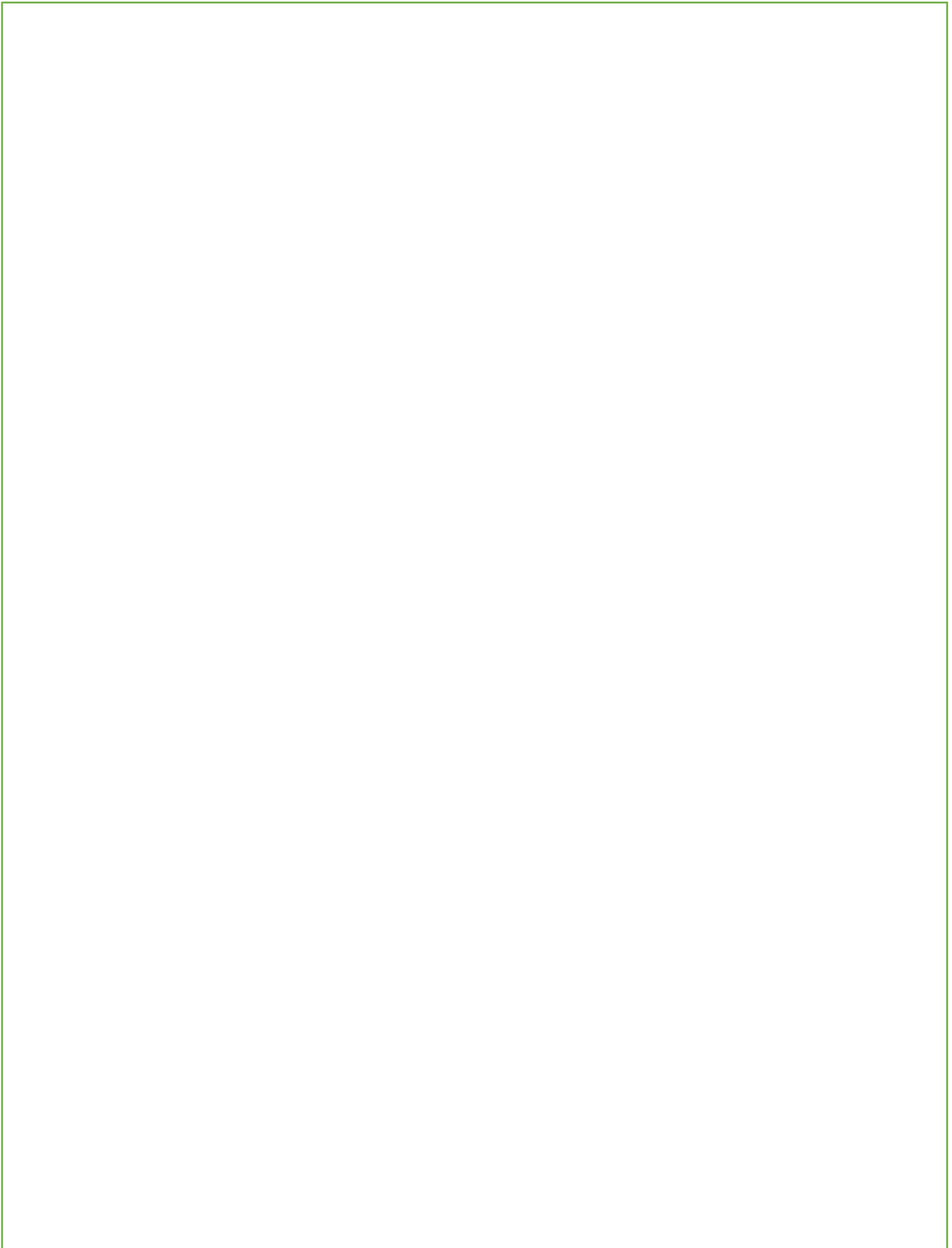
where a is a positive constant.

Sketch the following curves.

a $r = a(1 + \cos\theta)$ **b** $r = a\sin 3\theta$ **c** $r^2 = a^2\cos 2\theta$
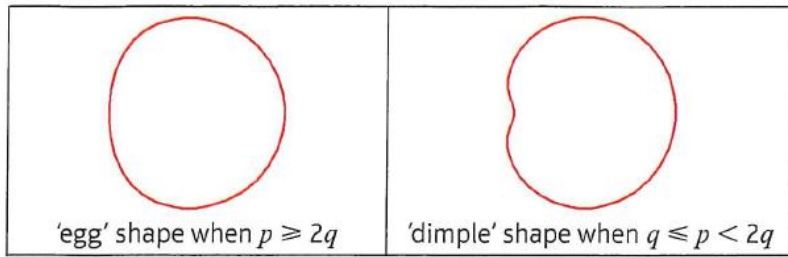
a $r = a(1 + \cos\theta)$ **b** $r = a\sin 3\theta$ **c** $r^2 = a^2\cos 2\theta$

Curves with equations of the form $r = a(p + q\cos\theta)$ are defined for all values of θ if $p \geqslant q$. An example of this, when $p = q$, was the cardioid seen in Example 6a. These curves fall into two types, those that are 'egg' shaped (i.e. a convex curve) and those with a 'dimple' (i.e. the curve is concave at $\theta = \pi$). The conditions for each type are given below:

'egg' shape when $p \geqslant 2q$

'dimple' shape when $q \leqslant p < 2q$

Example

Sketch the following curves.

a $r = a(5 + 2\cos\theta)$

b $r = a(3 + 2\cos\theta)$

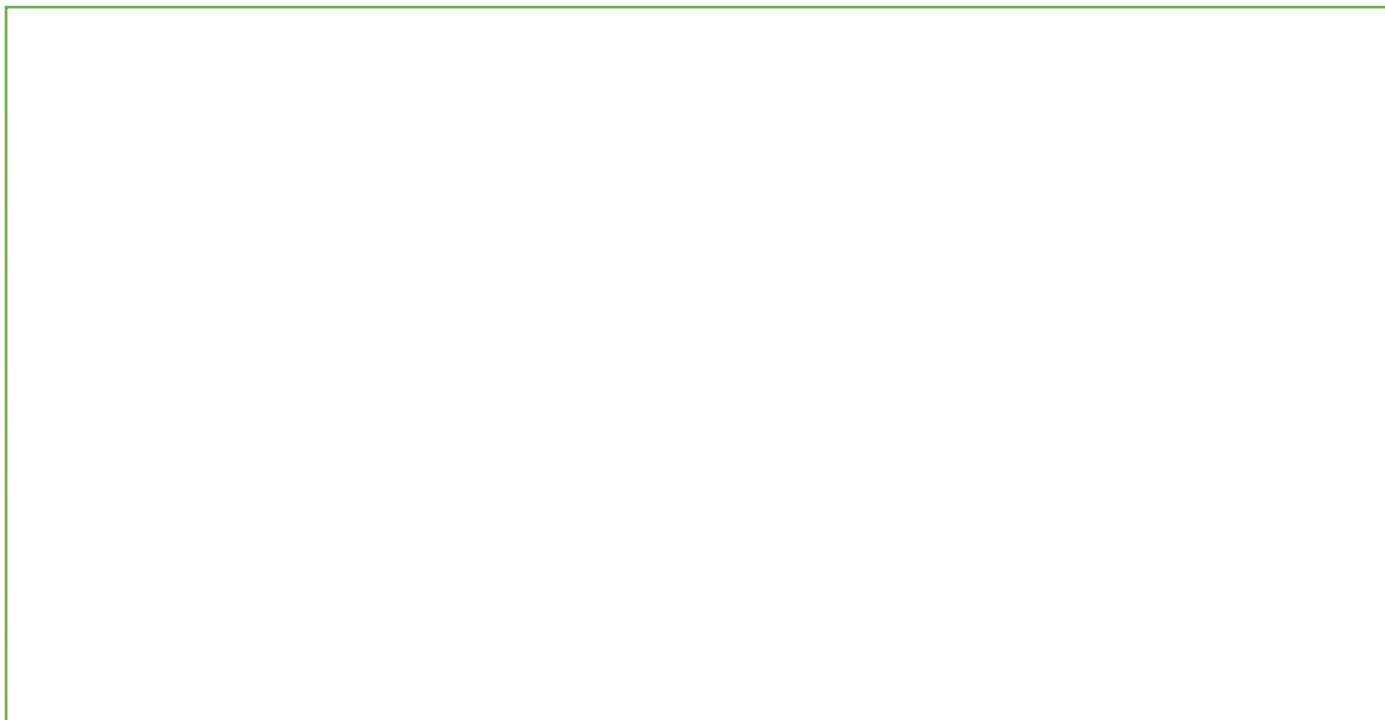You may also need to find a polar curve to represent a locus of points on an Argand diagram.

Example

a Show on an Argand diagram the locus of points given by the values of z satisfying
$|z - 3 - 4i| = 5$

b Show that this locus of points can be represented by the polar curve $r = 6\cos\theta + 8\sin\theta$.
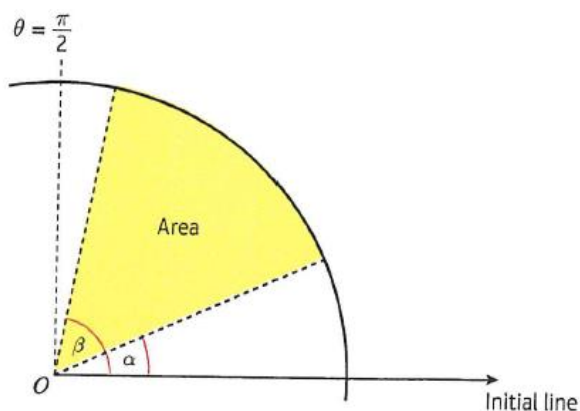
Area Enclosed By A Curve

You can find areas enclosed by a polar curve using integration.

■ **The area of a sector bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by the formula**

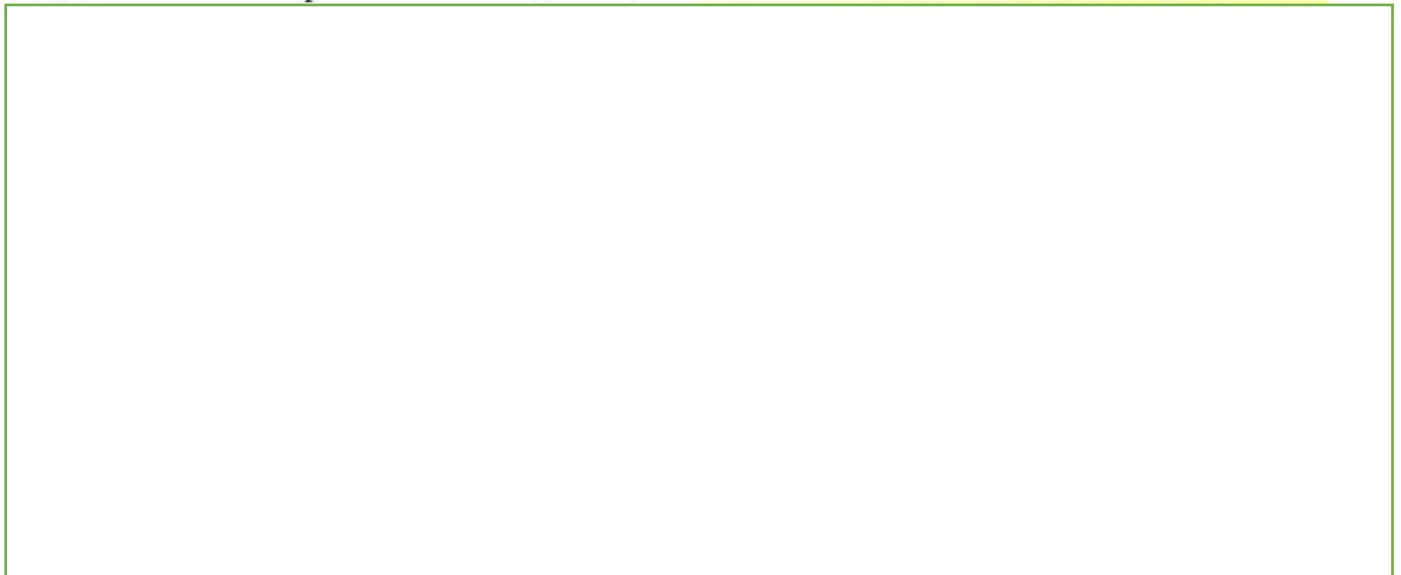$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Example

Find the area enclosed by the cardioid with equation $r = a(1 + \cos\theta)$.

Example

Find the area of one loop of the curve with polar equation $r = a\sin 4\theta$.
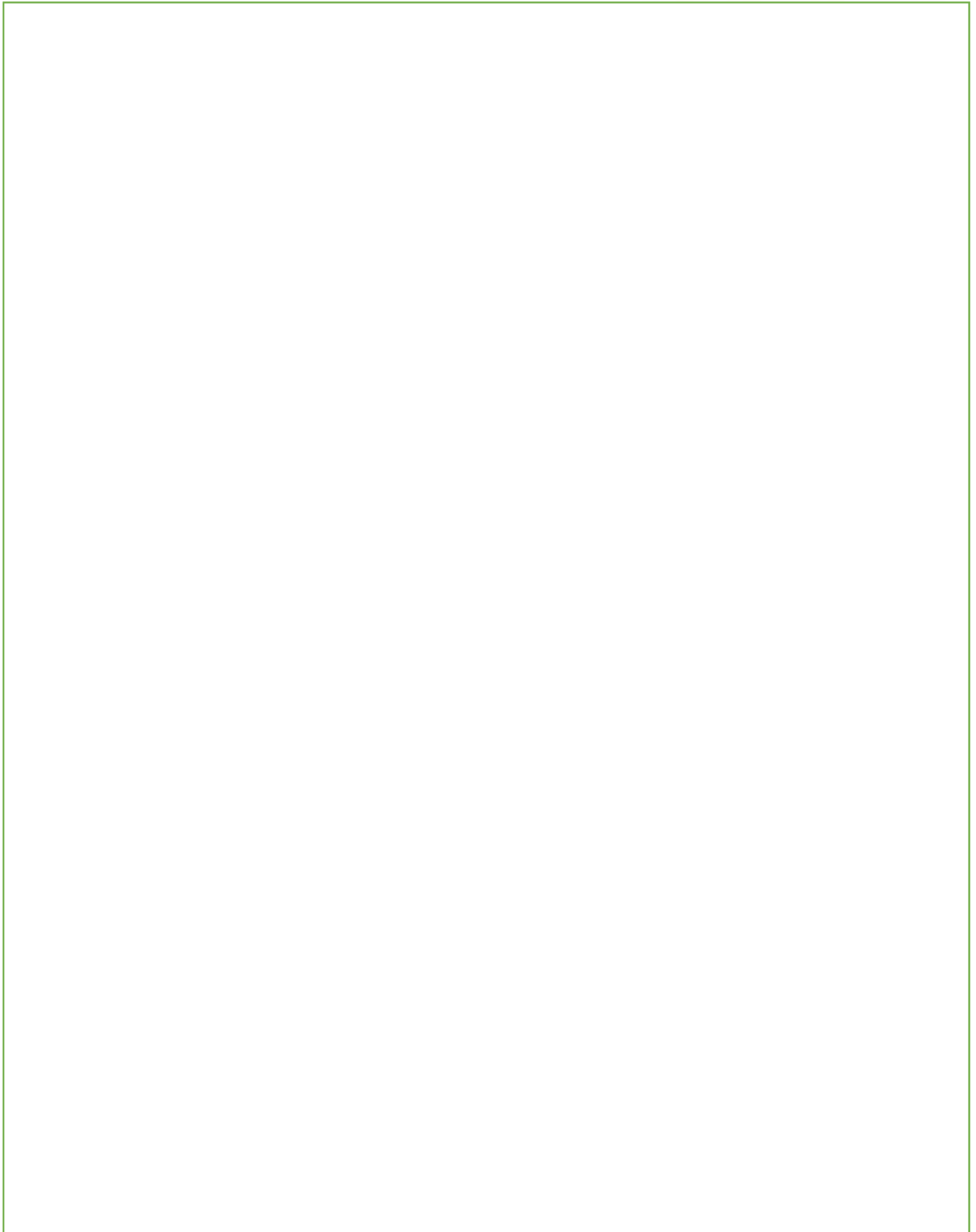
Example

a On the same diagram, sketch the curves with equations $r = 2 + \cos\theta$ and $r = 5\cos\theta$.

b Find the polar coordinates of the points of intersection of these two curves.

c Find the exact area of the region which lies within both curves.

If you are given a curve $r = f(\theta)$ in polar form, you can write it as a parametric curve in Cartesian form, using θ as the parameter:

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

By differentiating parametrically, you can find the gradient of the curve at any point:

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}$$

When $\dfrac{dy}{d\theta} = 0$, a tangent to the curve will be horizontal.

When $\dfrac{dx}{d\theta} = 0$, a tangent to the curve will be vertical.

You need to be able to find tangents to a polar curve that are **parallel** or **perpendicular** to the initial line.

- **To find a tangent parallel to the initial line set $\dfrac{dy}{d\theta} = 0$.**

- **To find a tangent perpendicular to the initial line set $\dfrac{dx}{d\theta} = 0$.**

Example

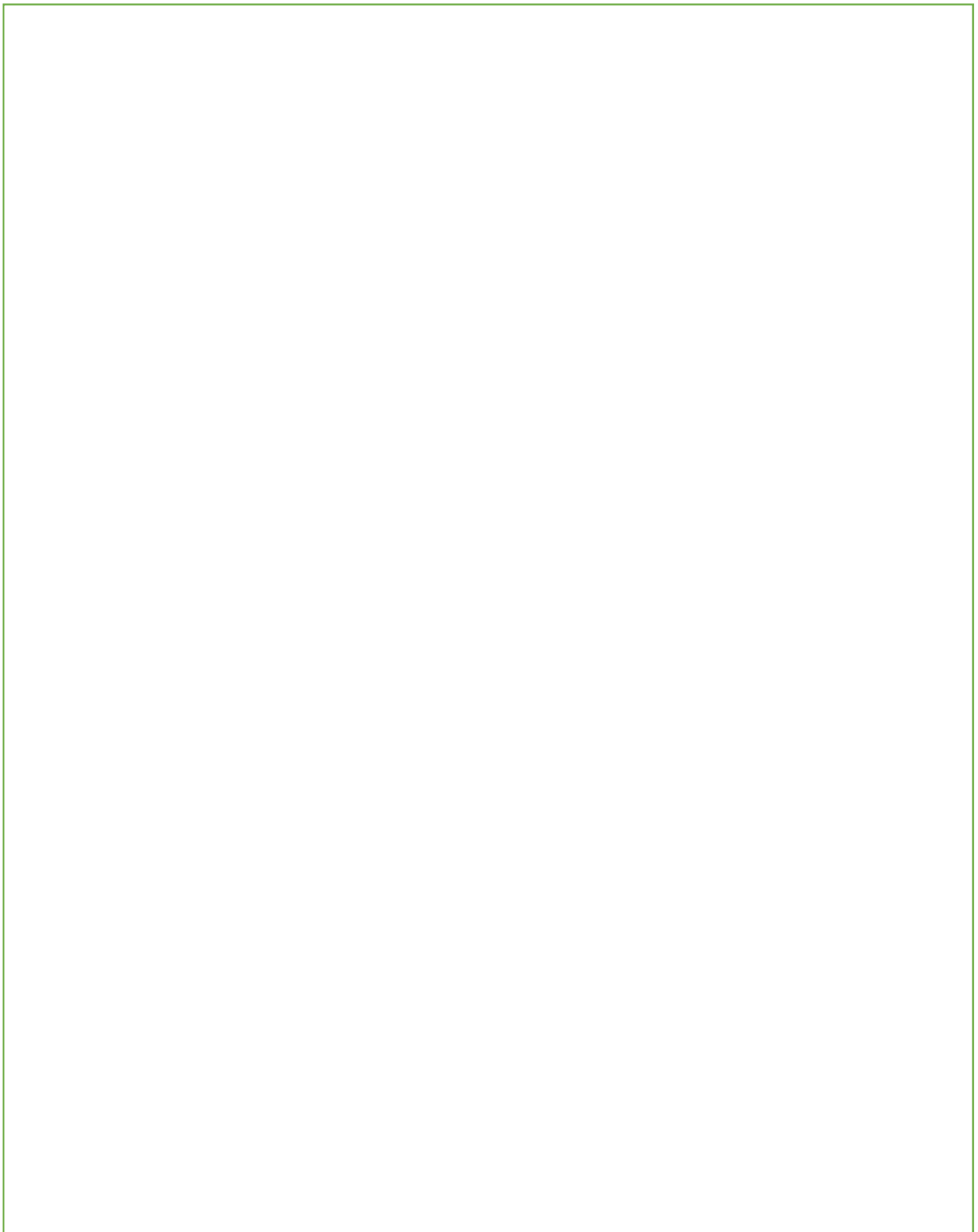Find the coordinates of the points on $r = a(1 + \cos\theta)$ where the tangents are parallel to the initial line $\theta = 0$.

Find the equations and the points of contact of the tangents to the curve $r = a\sin 2\theta$, $0 \leqslant \theta \leqslant \dfrac{\pi}{2}$ that are:

a parallel to the initial line

b perpendicular to the initial line.

Give answers to three significant figures where appropriate.

Summary of key points

1 For a point P with polar coordinates (r, θ) and Cartesian coordinates (x, y),
   - $r\cos\theta = x$ and $r\sin\theta = y$
   - $r^2 = x^2 + y^2$, $\theta = \arctan\left(\dfrac{y}{x}\right)$

   Care must be taken to ensure that θ is in the correct quadrant.

2 - $r = a$ is a circle with centre O and radius a.
   - $\theta = \alpha$ is a half-line through O and making an angle α with the initial line.
   - $r = a\theta$ is a spiral starting at O.

3 The **area of a sector** bounded by a polar curve and the half-lines $\theta = \alpha$ and $\theta = \beta$, where θ is in radians, is given by the formula

$$\textbf{Area} = \tfrac{1}{2}\int_{\alpha}^{\beta} r^2 \, \textbf{d}\theta$$

4 - To find a tangent parallel to the initial line set $\dfrac{dy}{d\theta} = 0$.

   - To find a tangent perpendicular to the initial line set $\dfrac{dx}{d\theta} = 0$.