Finding the general equation of a first order differential equation in which the variables are separable.

$$\frac{dy}{dx} = f(x)g(y)$$

$$\therefore \frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\therefore \int \frac{1}{g(y)}dy = \int f(x)dx + c$$

Example

Given that $y = 2$ at $x = 0$ and $\frac{dy}{dx} = y^2 + 4$, find y in terms of x.

Solution

*P3 book Ex8A Q18-22

First Order Linear Differential Equaations

A 1st order linear differential equation is of the form

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of *x* or constants.

An equation of this form is said to be exact when one side is the exact derivative of a product and the other side can be integrated wrt *x*.

If it is not exact then it can be made exact by multiplying through the equation by a function of *x.* This function is called the integrating factor.

Example

Consider the differential equation

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

Multiplying through by *x* gives…

$$x\frac{dy}{dx} + y = x^3$$

…making it exact.

$$\therefore \frac{d(xy)}{dx} = x^3$$

$$\therefore xy = \int x^3 dx$$

$$\therefore xy = \frac{x^4}{4} + c$$

In this case the integrating factor is *x*.

Note:- The integrating factor is given by $f(x)$ where $f(x) = e^{\int P dx}$.

i.e. in the last example $P = \frac{1}{x}$

$$\therefore f(x) = e^{\int \frac{1}{x} dx}$$

$$\therefore f(x) = e^{\ln x}$$

$$\therefore f(x) = x$$

So the linear equation $\frac{dy}{dx} + Py = Q$ can be solved by multiplying by the integgrating factor $e^{\int Pdx}$, provided $e^{\int Pdx}$ can be found and the function $Qe^{\int Pdx}$ can be integrated wrt *x*.

Example

Find the general solution of the differential equation

$$\cos x \frac{dy}{dx} + y \sin x = \sin x\, cos^3 x$$

Solution

Example

Find y in terms of x given that

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \ln x \ \ for\ x > 0\ and\ y = 2\ at\ x = 1$$

Solution

*P3 book Ex8C Q1-9,13,14,16,17,18

The Second Order Linear Differential Equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \; where\; a, b\; and\; c\; are\; constants$$

The equation is called the 2nd order, because its highest derivative of y wrt x is $\frac{d^2y}{dx^2}$.

The equation is called linear because only 1st degree terms in y and its derivatives occur.

Result: The general solution of the 2nd order differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ is $y = Au + Bv$,

where $y = u \; and \; y = v$ are particular, distinct solutions of the differential equation.

We now need to find the functions $u \; and \; v$ in specific cases.

In the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, try as a solution…

$$y = e^{mx} \text{ where } m \text{ is a constant to be found.}$$

$$\frac{dy}{dx} = me^{mx}$$

$$\frac{d^2y}{dx^2} = m^2e^{mx}$$

If $y = e^{mx}$ is a solution of the differential equation then

$$am^2e^{mx} + b\,me^{mx} + ce^{mx} = 0$$

$$\therefore am^2 + b\,m + c = 0 \; (because\; e^{mx} > 0\; for\; all\; m)$$

The 2 values of m required are the roots of the quadratic equation $am^2 + b\,m + c = 0$. This equation is called the Auxiliary Quadratic Equation and it may have..

(i) Real roots $(if\; b^2 - 4ac > 0)$
(ii) Identical roots $(if\; b^2 - 4ac = 0)$
(iii) Complex roots $(if\; b^2 - 4ac < 0)$

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

Solution

Generalising:- The general solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, whose auxiliary quadratic equation $am^2 + b\,m + c = 0$ has real distinct roots $\alpha\ and\ \beta$ is:-

$$y = Ae^{\alpha x} + Be^{\beta x}$$

(where A and B are constants)

*P3 Book Ex8D

Auxiliary Quadratic Equation With Real Coincident Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Solution

Generalising:- The general solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, whose auxiliary quadratic equation $am^2 + b\,m + c = 0$ has equal roots α is:-

$$y = (A + Bx)e^{\alpha x}$$

(where A and B are constants)

*P3 Book Ex8E

Auxiliary Quadratic Equation With Pure Imaginary Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4y = 0$$

Solution

Result:- For the differential equation

$$\frac{d^2y}{dx^2} + n^2y = 0$$

General solution is $y = A\cos nx + B\sin nx$ (where A and B are constants).

Auxiliary Quadratic Equation With Complex Conjugate Roots

Example

Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$$

Solution

Result:- For the differential equation

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, where the auxiliary quadratic equation

$am^2 + b\,m + c = 0$ has complex conjugate roots

$p + iq \; and \; p - iq \; (where \; p \; and \; q \; \in R)$

the general solution is $y = e^{Px}(A\cos qx + B\sin qx)$

(where A and B are constants)

*P3 book Ex8F

The Second Order Differential Equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

To solve this type of differential equation:-

Method:-

1. Solve the differential equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

   The solution is called the *complementary function*.

2. Find a solution of the equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

$$where\ f(x) could\ be\ any\ one\ of\ these\ forms:-$$

   (i) A constant k

   (ii) A linear function $px+q$

   (iii) An exponential function ke^{px}

   (iv) A trig function e.g. $p\sin x, q\cos 2x\ or\ psin\ 3x + qcos3x$

A solution of the differential equation for any of the forms of $f(x)$ given above can be found by inspection.

This solution, when found, is called a *particular integral* of the equation.

3. The general solution of the diifferential equation is then

$$y=C.F.+P.I.$$

Examples on finding the P.I.

Example

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = f(x)$$

Find P.I. of this differential equation in the cases where $f(x) = \cdots$

(a.) 12 (b.) $3x + 5$ (c.) $3e^{2x}$ (d.) $\cos 2x$

Solution

(a.)

(b.)

(c.)

(d.)

Example

Find y in terms of x for the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \cos 2x$$

given that $\frac{dy}{dx} = 0\ at\ x = 0\ and\ y = 0\ at\ x = 0.$

Solution

*P3 book Ex8G Q1-9odds,17,20,23,26-30