# Further Maths AS1 Question Booklet

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# Further Maths AS1
# Matrices

***Exercise 6A*** Questions 4,6-10

**4.** Find the values of $x$ and $y$ in each of the following matrix equations.

(a) $\begin{pmatrix} 3 & -5 \\ 2 & x \end{pmatrix} + \begin{pmatrix} 1 & y \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 5 & -2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ y \end{pmatrix} - \begin{pmatrix} x \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$

(c) $2\begin{pmatrix} 3 \\ y \end{pmatrix} + \begin{pmatrix} x \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ -1 \end{pmatrix} + x\begin{pmatrix} -2 \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ 11 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & x \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 1 & y \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 13 & 7 \end{pmatrix}$ (f) $\begin{pmatrix} 2 & 1 \\ x & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 3 & y \end{pmatrix}$

(g) $\begin{pmatrix} 2 & x \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ y & -3 \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ 2 & 6 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

(i) $\begin{pmatrix} -1 & 3 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \end{pmatrix}$ (j) $\begin{pmatrix} 2 & 4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

**5.** By letting $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ prove that $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ and that $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

**6.** If $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 6 & 1 \\ 11 & 3 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -4 & 3 \\ -5 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 4 & 7 \\ -2 & 7 \end{pmatrix}$ find the $2 \times 2$ matrices **X**, **Y** and **Z** given that $\mathbf{AX} = \mathbf{B}$, $\mathbf{BY} = \mathbf{C}$ and $\mathbf{CZ} = \mathbf{D}$.

**7.** If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -4 & 1 \end{pmatrix}$ find the values of $m$ and $n$ given that $\mathbf{A}^2 = m\mathbf{A} + n\mathbf{I}$ where **I** is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

**8.** Find the possible values $x$ can take given that $\mathbf{A} = \begin{pmatrix} x^2 & 3 \\ 1 & 3x \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 6 \\ 2 & x \end{pmatrix}$ and $\mathbf{AB} = \mathbf{BA}$.

**9.** Solve the following simultaneous equations by matrix methods.

(a) $x - y = 5$, $3x + 2y = 5$ (b) $x - 3y = 3$, $5x - 9y = 11$ (c) $x + 3y = 1$, $2x - 4y = 1$

**10.** If $\mathbf{A} = \begin{pmatrix} -3 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -4 & -3 & 5 \\ -5 & -4 & 7 \\ 1 & 1 & -1 \end{pmatrix}$ find **AB**.

Hence find the values of $x$, $y$ and $z$ satisfying the three equations

$$-4x - 3y + 5z = 3$$
$$-5x - 4y + 7z = 4$$
$$x + y - z = 0.$$

**Answers**

**4.** (a) $-4, 11$ (b) $-2, -3$ (c) $2, 4$ (d) $4, 3$ (e) $2, -1$ (f) $5, 19$ (g) $-1, 4$ (h) $2, -1$ (i) $-3, 2$ (j) $-3, 1\frac{1}{2}$

**6.** $\mathbf{X} = \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}$ $\mathbf{Y} = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$ $\mathbf{Z} = \begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}$ **7.** $4, -11$ **8.** $-\frac{1}{2}$ or $3$

**9.** (a) $x = 3, y = -2$ (b) $x = 1, y = -\frac{2}{3}$ (c) $x = 1\frac{1}{2}, y = \frac{1}{2}$ **10.** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $-1, 2, 1$.

**Exercise 6B** **Questions 1,3,5-10,13**

1. The points A(3, 2), B(−1, 4), C(2, 5) and D(1, −1) are transformed to A′, B′, C′, and D′ by the transformation matrix $\begin{pmatrix} 3 & 0 \\ 1 & -3 \end{pmatrix}$. Find the coordinates of A′, B′, C′, and D′.

2. Under a certain transformation the image $(x', y')$ of a point $(x, y)$ is obtained by the rule:

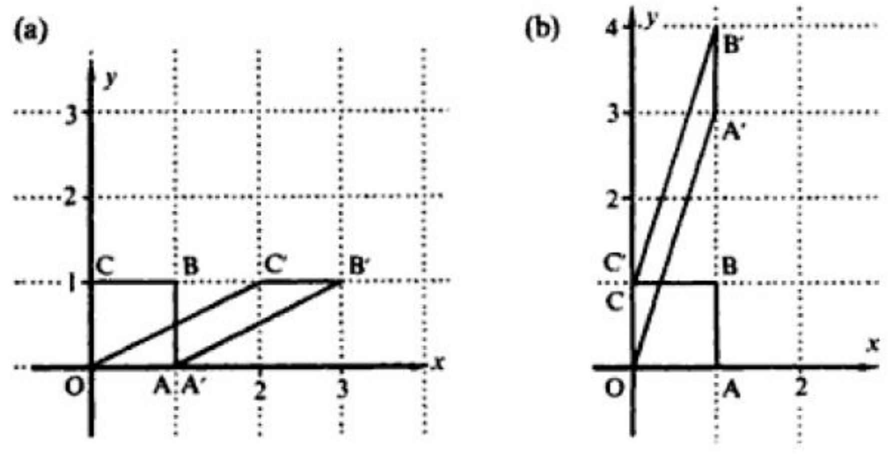
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Find (a) the image of the point (2, 1), (b) the image of the point (−1, 3), (c) the point with an image of (1, −6), (d) the point with an image of (−1, 11).

3. Find the 2 × 2 transformation matrix that will map the point (−2, 3) onto (−7, 6) and (1, −1) onto (3, −1).
Find the image of the point (−1, 3) under this linear transformation and the coordinates of the point that has an image of (6, −2).

5. Find the 2 × 2 transformation matrices **P**, **Q**, **R** and **S** given that
(a) matrix **P** represents a reflection in the $y$-axis,
(b) matrix **Q** represents a 90° clockwise rotation about the origin,
(c) matrix **R** represents a reflection in the line $y = x$,
(d) matrix **S** transforms the point (2, −3) to (4, 14) and the point (1, 3) to (11, −2).
Use your answers to (a), (b) and (c) to show that a reflection in the $y$-axis followed by a 90° clockwise rotation about the origin is equivalent to a reflection in the line $y = x$.

6. The linear transformations shown below are shears transforming OABC to OA′B′C′. For each transformation, write down
(i) the associated 2 × 2 matrix,
(ii) the equations of the transformation in the form $x' = ax + by$
$y' = cx + dy$.




7. A 1st shape is transformed to a 2nd shape by the transformation matrix $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$ and a 3rd shape is obtained from the 2nd shape using the transformation matrix $\begin{pmatrix} -1 & 5 \\ -1 & 2 \end{pmatrix}$. Find the single matrix that would transform the 1st shape to the 3rd shape direct. If the 1st shape has an area of 5 sq. units, find the areas of the 2nd and 3rd shapes.

8. Find the matrices corresponding to the following linear transformations.
(a) 180° rotation about the origin,
(b) enlargement scale factor 3, centre (0, 0),
(c) reflection in the line $y = -x$,
(d) stretch (×2) parallel to the $y$-axis, $x$-axis fixed,
(e) shear with $x$-axis fixed and (0, 1) → (1, 1),
(f) shear with $y = x$ fixed and (1, 0) → (0, −1),
(g) stretch (×3) perpendicular to $y = -x$ and with $y = -x$ fixed,
(h) shear with $y = x$ fixed and (0, 2) → (4, 6),
(i) shear with $y = 2x$ fixed and (0, 4) → (−2, 0).

**9.** Give a geometrical description of the effect of each of the following transformation matrices.

(a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ (i) $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$ (j) $\begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$

**10.** Prove that the transformation matrix $\begin{pmatrix} 3 & 6 \\ -2 & -4 \end{pmatrix}$ maps all points of the $x$–$y$ plane onto a straight line and find the equation of that line.

**13.** A certain transformation maps a point A$(x, y)$ onto its image A$'(x', y')$ according to the rule:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

Find (a) the image of the point $(2, -1)$,

(b) the point with an image of $(-7, 6)$,

(c) the coordinates of the point that is mapped onto itself by the transformation.

**Answers**

**1.** $(9, -3), (-3, -13), (6, -13), (3, 4)$ **2.** (a) $(4, 1)$ (b) $(5, 10)$ (c) $(3, -1)$ (d) $(-5, 2)$

**3.** $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $(-5, 9)$, $(2, -2)$

**4.** (a) $(5, -3), (15, -9), (13, -7), (3, -1)$ (b) 8 sq. units (c) $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$

**5.** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 5 & 2 \\ 4 & -2 \end{pmatrix}$

**6.** (a) (i) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{matrix} x' = x + 2y \\ y' = y \end{matrix}$ (b) (i) $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ (ii) $\begin{matrix} x' = x \\ y' = 3x + y \end{matrix}$

**7.** $\begin{pmatrix} 18 & 14 \\ 6 & 5 \end{pmatrix}$, 10 sq. units, 30 sq. units

**8.** (a) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ (e) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ (g) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ (i) $\begin{pmatrix} 2 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$

**9.** (a) reflection in $y = x$ (b) enlargement ($\times$ 5) centre $(0, 0)$
(c) stretch ($\times$ 3) parallel to $x$-axis, $y$-axis fixed (d) mapping onto $(0, 0)$
(e) mapping onto $x$-axis (f) mapping onto $y = x$
(g) enlargement ($\times$ 3) centre $(0, 0)$ and reflection in $y = x$
(h) shear with $y$-axis fixed and $(1, 0) \to (1, 2)$ (i) mapping onto $y = 4x$
(j) shear with $y = x$ fixed and $(1, 0) \to (-2, -3)$

**10.** $2x + 3y = 0$ **12.** (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (b) $\begin{matrix} x' = -x + 2 \\ y' = y + 3 \end{matrix}$

**13.** (a) $(-1, -1)$ (b) $(-3, 2)$ (c) $(-1, 2)$

**Exercise 6C Questions 1,3,4,5,6,8,10,13,14,17**

**1.** Find any invariant points of the transformations given by

(a) $x' = 3y + 2$
$y' = 2x - y + 4$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

**2.** Two transformations **P** and **T** transform the point $(x, y)$ to its image $(x', y')$ according to the following rules:

$\mathbf{P}\begin{cases} x' = 3x + y \\ y' = x - 1 \end{cases}$ $\mathbf{T}\begin{cases} x' = 2x - y - 3 \\ y' = 1 - x \end{cases}$

Express the transformations **P**, **T**, **PT** and **TP** in the form

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} * \\ * \end{pmatrix}$

Find any invariant points under the transformation
(a) **P**, (b) **T**, (c) **PT**, (d) **TP**.

**3.** Find the equations of the image lines formed when the lines $y = 2x + 1$ and $3y = 2x - 1$ are transformed using the matrix $\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}$.

**4.** Find the equation of the image line produced by translating all of the points on the line $y = 3x - 1$ by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

**5.** Find vector equations of the image lines formed when the lines $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ are transformed using the matrix $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$.

**6.** Find the $2 \times 2$ transformation matrix **T** that maps $(3, -1)$ onto $(13, -7)$ and $(-1, 3)$ onto $(1, 5)$. Find the equations of the lines obtained when **T** is applied to $y = x$, $y + 2x = 3$ and $y = 2x + 3$.

**7.** The transformation matrix $\begin{pmatrix} 3 & -2 \\ 3 & -1 \end{pmatrix}$ transforms a line **L** to the line $y = 2x + 3$. Find the equation of **L**.

**8.** Find the vector equation of the line which, when transformed by the matrix $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, has an image line $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

**9.** A transformation **T** assigns to any point $(x, y)$ an image $(x', y')$ according to the rule $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ -4 \end{pmatrix}$.

(a) Find the equation of the image lines obtained when all points on the lines $y = 2x$ and $y = x - 3$ undergo the transformation **T**.
(b) Prove that the line $y = 3x + 1$ maps onto itself under the transformation **T**.

**10.** Prove that the transformation matrix $\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$ maps all points on the line $y = 3x$ onto themselves.

**11.** Prove that all lines of the form $y = x + a$ are mapped onto themselves under the transformation given by the matrix $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$.

**12.** Prove that the lines $y = x + 5$ and $y + 3x = 1$ are mapped onto themselves under the transformation that maps $(x, y)$ onto $(x', y')$ according to the relationship $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -9 \end{pmatrix}$.

**13.** Show that the transformation with matrix $\begin{pmatrix} 1 & 3 \\ -1 & -3 \end{pmatrix}$ maps all points on the line $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ onto a single point and find the position vector of this point.

**14.** Show that the transformation with matrix $\begin{pmatrix} -4 & 2 \\ 6 & -3 \end{pmatrix}$ maps all points on the line $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \lambda(\mathbf{i} + 2\mathbf{j})$ onto a single point and find the position vector of this point.

**15.** Prove that all points on the line $y + 3x + 2 = 0$ are mapped onto a single point under the transformation that maps $(x, y)$ onto $(x', y')$ according to the relationship $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and find the coordinates of this single point.

**16.** Find the equations of any straight lines that pass through the origin and that map onto themselves under the transformation defined by the matrix

(a) $\begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 3 \\ 1 & -5 \end{pmatrix}$ (c) $\begin{pmatrix} a & 1 \\ 8 & a \end{pmatrix}$

**17.** Find the $2 \times 2$ matrix that maps $(1, 2)$ onto $(-3, 0)$ and $(-2, -3)$ onto $(2, -1)$. For the transformation defined by this matrix, find the equations of two invariant straight lines passing through the origin. Which of these two lines is a set of invariant points under the transformation?

**Answers**

**1.** (a) $(-4, -2)$ (b) $(-3, 2)$ (c) $(4\frac{1}{2}, 1)$

**2.** $\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 5 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -8 \\ -4 \end{pmatrix}$, $\begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(a) $(\frac{1}{3}, -\frac{2}{3})$ (b) $(2, -1)$ (c) $(2, 0)$ (d) $(1, -1)$

**3.** $7y = 5x + 6$, $x = 3y + 2$ **4.** $y = 3x - 4$ **5.** $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 11 \\ 1 \end{pmatrix}$, $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 5 \end{pmatrix}$

**6.** $\begin{pmatrix} 5 & 2 \\ -2 & 1 \end{pmatrix}$, $x + 7y = 0$, $4x + y = 27$, $y = 3$ **7.** $y = x + 1$ **8.** $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

**9.** (a) $3y = 10x + 8$, $y = 5x + 21$ **13.** $\begin{pmatrix} 7 \\ -7 \end{pmatrix}$ **14.** $-2\mathbf{i} + 3\mathbf{j}$ **15.** $(-1, 1)$

**16.** (a) $y = x$, $y = 2x$ (b) $3y = x$, $y + x = 0$ (c) $y = \pm 2\sqrt{2}x$

**17.** $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$, $y = x$, $2y = x$; $y = x$ **18.** $8$, $y = 2x$

**19.** (a) $y = x + 6$, $3y = 2x + 12$ (b) $y + 4x = 1$, $y = x + 1$ (c) $y = 2$, $y = 2x - 6$ (d) $y = x - 1$ (e) $y = 3x - 9$

**Exercise 6D** **Questions 1a,2a,3,4**

1. Find the matrices which represent the following linear transformations.
   (a) a rotation of 30° anticlockwise about the origin,
   (b) a rotation of 45° anticlockwise about the origin,
   (c) a rotation of 120° anticlockwise about the origin.
2. Find the matrices which represent the following linear transformations,
   (a) a reflection in the line $y = \sqrt{3}x$,
   (b) a reflection in the line $\sqrt{3}y = x$,
   (c) a reflection in the line $y = 2x$.
3. Give a geometrical description of the transformations corresponding to the following matrices

   (a) $\begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$ (c) $\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$
4. Find the matrix equations representing each of the following transformations,
   (a) a rotation of 90° anticlockwise about the point $(-1, 4)$,
   (b) a rotation of 180° about the point $(3, -1)$,
   (c) an enlargement, scale factor 3, centre $(2, -1)$,
   (d) a reflection in the line $y = x + 5$,
   (e) a glide reflection in the line $y = x + 5$ with the point $(0, 5)$ mapped onto $(3, 8)$.

**Answers**

1. (a) $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (c) $\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

2. (a) $\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ (c) $\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

3. (a) rotation of 53·1° anticlockwise about origin (b) reflection in $2y = x$
   (c) rotation of 53·1° anticlockwise about origin and enlargement (× 5), centre the origin

4. (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

   (c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ (d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 5 \end{pmatrix}$

   (e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 8 \end{pmatrix}$

**P4 Book Ex4B Questions 3, 4alt**

3 Evaluate:

(a) $\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$ (d) $\begin{vmatrix} 0 & -2 \\ 1 & -4 \end{vmatrix}$

✓ (e) $\begin{vmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ -1 & 4 & -6 \end{vmatrix}$ ✓ (f) $\begin{vmatrix} -2 & 7 & 3 \\ 1 & 2 & 4 \\ -1 & 2 & 0 \end{vmatrix}$

(g) $\begin{vmatrix} 2 & -3 & 6 \\ -2 & 4 & 5 \\ -1 & 0 & -5 \end{vmatrix}$ (h) $\begin{vmatrix} 1 & 2 & -3 \\ 2 & 2 & -4 \\ -4 & 2 & 1 \end{vmatrix}$

(i) $\begin{vmatrix} -3 & 1 & 2 \\ 0 & 1 & -3 \\ 3 & 3 & -2 \end{vmatrix}$ (j) $\begin{vmatrix} 3 & 4 & 1 \\ 4 & 1 & 3 \\ 2 & 5 & 2 \end{vmatrix}$

(k) $\begin{vmatrix} 1 & -1 & 3 \\ 2 & -2 & 4 \\ 3 & -3 & 5 \end{vmatrix}$ (l) $\begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 7 \\ -2 & -7 & 0 \end{vmatrix}$

(m) $\begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{vmatrix}$ (n) $\begin{vmatrix} 3 & 0 & 2 \\ 0 & -1 & 4 \\ 1 & 1 & -2 \end{vmatrix}$

(o) $\begin{vmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \\ 1 & 0 & 2 \end{vmatrix}$ (p) $\begin{vmatrix} 5 & 1 & 3 \\ -2 & 0 & 1 \\ 1 & 1 & -2 \end{vmatrix}$

4 Find the inverse of:

(a) $\begin{pmatrix} 2 & 5 \\ -1 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -3 & 2 \\ -1 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -3 \\ 1 & -4 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 1 \\ -3 & 2 \end{pmatrix}$

✓ (e) $\begin{pmatrix} -3 & 7 \\ 9 & 22 \end{pmatrix}$ ✓ (f) $\begin{pmatrix} 2 & 1 & -5 \\ 1 & 0 & -2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} -1 & 2 & 1 \\ 0 & 0 & -2 \\ 1 & -5 & 4 \end{pmatrix}$ (h) $\begin{pmatrix} -1 & 2 & 3 \\ 1 & 1 & 2 \\ 5 & -1 & 4 \end{pmatrix}$

✓ (i) $\begin{pmatrix} -2 & 3 & -4 \\ 1 & 2 & -3 \\ -3 & 0 & -2 \end{pmatrix}$ (j) $\begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{pmatrix}$

(k) $\begin{pmatrix} 3 & 2 & -6 \\ 1 & 1 & -2 \\ 2 & 2 & -1 \end{pmatrix}$ ✓ (l) $\begin{pmatrix} 4 & -5 & 2 \\ 0 & 1 & -7 \\ 1 & 1 & -2 \end{pmatrix}$

(m) $\begin{pmatrix} 3 & 2 & -3 \\ 1 & 1 & -4 \\ 2 & 2 & -6 \end{pmatrix}$ (n) $\begin{pmatrix} -2 & 1 & -2 \\ 4 & 3 & 1 \\ 0 & 1 & -6 \end{pmatrix}$

✓ (o) $\begin{pmatrix} 2 & 1 & -3 \\ 1 & -1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$ ✓ (p) $\begin{pmatrix} 2 & 2 & 1 \\ 4 & 1 & 5 \\ -1 & 1 & 7 \end{pmatrix}$

**Answers**

3 (a) 1 (b) $-2$ (c) $-5$ (d) 2
(e) $-9$ (f) 0 (g) 29 (h) 2
(i) $-36$ (j) $-29$ (k) 0 (l) 0
(m) 0 (n) $-4$ (o) 6 (p) $-14$

4 (a) $\frac{1}{13}\begin{pmatrix} 4 & -5 \\ 1 & 2 \end{pmatrix}$

(b) $-\frac{1}{19}\begin{pmatrix} 7 & -2 \\ 1 & -3 \end{pmatrix}$

(c) $-\frac{1}{5}\begin{pmatrix} -4 & 3 \\ -1 & 2 \end{pmatrix}$

(d) $\frac{1}{3}\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$

(e) $-\frac{1}{129}\begin{pmatrix} 22 & -7 \\ -9 & -3 \end{pmatrix}$

(f) $-\frac{1}{3}\begin{pmatrix} 0 & -3 & -2 \\ -3 & 6 & -1 \\ 0 & 0 & -1 \end{pmatrix}$

(g) $\frac{1}{6}\begin{pmatrix} -10 & -13 & -4 \\ -2 & -5 & -2 \\ 0 & -3 & 0 \end{pmatrix}$

(h) $-\frac{1}{12}\begin{pmatrix} 6 & -11 & 1 \\ 6 & -19 & 5 \\ -6 & 9 & -3 \end{pmatrix}$

(i) $\frac{1}{17}\begin{pmatrix} -4 & 6 & -1 \\ 11 & -8 & -10 \\ 6 & -9 & -7 \end{pmatrix}$

(j) $\frac{1}{5}\begin{pmatrix} 2 & -1 & 1 \\ 5 & -10 & 0 \\ 3 & -4 & -1 \end{pmatrix}$

(k) $\frac{1}{3}\begin{pmatrix} 3 & -10 & 2 \\ -3 & 9 & 0 \\ 0 & -2 & 1 \end{pmatrix}$

(l) $\frac{1}{53}\begin{pmatrix} 5 & -8 & 33 \\ -7 & -10 & 28 \\ -1 & -9 & 4 \end{pmatrix}$

**P4 Book Ex4C Questions 1,4,5,8**

**1** Given that

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

$$U : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 2x+y \end{pmatrix}$$

find the matrix that represents (a) $T$ (b) $U$ (c) $TU$ (d) $UT$.

**4** Given that

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

$$U : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x \\ y \end{pmatrix}$$

find the matrix that represents (a) $T^{-1}$ (b) $U^{-1}$ (c) $(UT)^{-1}$.

**5** Given that

$$T : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2x+z \\ y \\ -y+z \end{pmatrix}$$

$$U : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} -x+2y-3z \\ 2x-y+4z \\ 3x+4y+z \end{pmatrix}$$

find the matrix that represents (a) $T$ (b) $U$ (c) $TU$ (d) $UT$.

**8** Given that

$$T : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x-y+3z \\ 2x+y+4z \\ y+z \end{pmatrix}$$

$$U : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+3y-2z \\ -2x-9y+5z \\ x+10y+4z \end{pmatrix}$$

find the matrix that represents
(a) $T^{-1}$ (b) $U^{-1}$ (c) $(TU)^{-1}$ (d) $(UT)^{-1}$.

**9** Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 1 & 7 \\ -2 & 3 & 6 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -2 & 1 & 3 \\ 4 & -9 & 5 \\ 1 & 1 & -2 \end{pmatrix}$$

find (a) $\mathbf{A}^{\mathrm{T}}$ (b) $\mathbf{B}^{\mathrm{T}}$.
Hence find (c) $(\mathbf{AB})^{\mathrm{T}}$ (d) $(\mathbf{BA})^{\mathrm{T}}$.

**10** Given that

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 4 & -7 \\ 6 & 6 & -3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & -2 \\ -1 & 1 & 3 \end{pmatrix}$$

find (a) $\mathbf{A}^{\mathrm{T}}$ (b) $\mathbf{B}^{\mathrm{T}}$.
Hence find (c) $(\mathbf{AB})^{\mathrm{T}}$ (d) $(\mathbf{BA})^{\mathrm{T}}$.

**Answers**

**1** (a) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 1 \\ -1 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$

**4** (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$

**5** (a) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 2 & -3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 8 & -5 \\ 2 & -1 & 4 \\ 1 & 5 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} -2 & 5 & -4 \\ 4 & -5 & 6 \\ 6 & 3 & 4 \end{pmatrix}$

**6** (a) $\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

(b) $\frac{1}{4}\begin{pmatrix} -17 & -14 & 5 \\ 10 & 8 & -2 \\ 11 & 10 & -3 \end{pmatrix}$

(c) $\frac{1}{8}\begin{pmatrix} -17 & -1 & 27 \\ 10 & 2 & -14 \\ 11 & 3 & -17 \end{pmatrix}$

(d) $\frac{1}{8}\begin{pmatrix} -38 & -32 & 10 \\ 20 & 16 & -4 \\ 42 & 36 & -10 \end{pmatrix}$

**7** (a) $\begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 & -2 \\ -2 & -9 & 5 \\ 1 & 10 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 6 & 42 & 5 \\ 4 & 37 & 17 \\ -1 & 1 & 9 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & 0 & 13 \\ -20 & -2 & -37 \\ 21 & 13 & 47 \end{pmatrix}$

**8** (a) $\frac{1}{5}\begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$

(b) $\frac{1}{25}\begin{pmatrix} 86 & 32 & 3 \\ -13 & -6 & 1 \\ 11 & 7 & 3 \end{pmatrix}$

(c) $-\frac{1}{125}\begin{pmatrix} 316 & -373 & 529 \\ -53 & 59 & -82 \\ 41 & -48 & 54 \end{pmatrix}$

(d) $-\frac{1}{125}\begin{pmatrix} 387 & 169 & 26 \\ 163 & 56 & -1 \\ -218 & -91 & -14 \end{pmatrix}$

**9** (a) $\begin{pmatrix} 1 & 4 & -2 \\ 0 & 1 & 3 \\ 2 & 7 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} -2 & 4 & 1 \\ 1 & -9 & 1 \\ 3 & 5 & -2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 3 & 22 \\ 3 & 2 & -23 \\ -1 & 3 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} -4 & -42 & 9 \\ 10 & 6 & -5 \\ 21 & -25 & -3 \end{pmatrix}$

**10** (a) $\begin{pmatrix} 2 & 1 & 6 \\ -1 & 4 & 6 \\ 3 & -7 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 4 & -2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} -4 & 20 & 27 \\ 6 & -1 & 15 \\ 19 & -25 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 28 & -5 & 17 \\ 31 & -11 & 23 \\ -23 & 8 & -19 \end{pmatrix}$

**System of Linear Equations Questions**

**Q1**

Consider the following system of simultaneous equations

$$\begin{aligned} x - y + 2z &= 6 \\ 2x + 3y - z &= 7 \\ x + 9y - 8z &= -4 \end{aligned}$$

i) By evaluating an appropriate determinant, show that this system does not have a unique solution.

ii) Solve this system of simultaneous equations. (NICCEA)

**Q2**

Consider the system of simultaneous equations

$$\begin{aligned} 3x + y - 2z &= -4 \\ x + 2y + 3z &= 11 \\ 3x - 4y - 13z &= -41 \end{aligned}$$

i) Solve this system of equations.

ii) Hence show in a sketch how the planes defined by the above equations are arranged so that the solution is of the form found in part i. (NICCEA)

**Q3**

Show that the equations

$$\begin{aligned} x + \lambda y + z &= 2a \\ x + y + \lambda z &= 2b \\ \lambda x + y + \lambda z &= 2c \end{aligned}$$

where $a, b, c \in \mathbb{R}$, have a unique solution for $x, y, z$ provided that $\lambda \neq 1$ and $\lambda \neq -1$.

a) In the case when $\lambda = 1$, state the condition to be satisfied by $a$, $b$ and $c$ for the equations to be consistent.

b) In the case when $\lambda = -1$, show that for the equations to be consistent

$$a + c = 0$$

Solve the equations in this case.

Give a geometrical description of the configuration of the three planes represented by the equations in the cases:

i) $\lambda = -1$ and $a + c = 0$

ii) $\lambda = -1$ and $a + c \neq 0$. (NEAB)

**Q4**

Find the values of $k$ for which the simultaneous equations

$$\begin{aligned} kx + 2y + z &= 0 \\ 3x - 2z &= 4 \\ 3x - 6ky - 4z &= 14 \end{aligned}$$

do not have a unique solution for $x$, $y$ and $z$.

Show that, when $k = -2$, the equations are inconsistent, and give a geometrical interpretation of the situation in this case. (OCR)

**Q5**

Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & -k \\ 1 & -k & -1 \end{pmatrix}$$

find $\det \mathbf{M}$ in terms of $k$.

Determine the values of $k$ for which the simultaneous equations

$$\begin{aligned} x+y-z&=1 \\ x+2y-kz&=0 \\ x-ky-z&=1 \end{aligned}$$

have a unique solution.

**i)** Solve these equations in the case when $k=2$.
**ii)** Show that the equations have no solution when $k=1$.
**iii)** Find the general solution when $k=-1$.

Give a geometrical interpretation of the equations in each of the three cases $k=2$, $k=1$ and $k=-1$. (NEAB)

**Q6**

Show that the only real value of $\lambda$ for which the simultaneous equations

$$\begin{aligned} (2+\lambda)x-y+z&=0 \\ x-2\lambda y-z&=0 \\ 4x-y-(\lambda-1)z&=0 \end{aligned}$$

have a solution other than $x=y=z=0$ is $-1$.

Solve the equations in the case when $\lambda=-1$, and interpret your result geometrically. (NEAB)

**Q7**

Consider the system of equations $x$, $y$ and $z$,

$$\begin{aligned} 2x+3y-z&=p \\ x-2z&=-5 \\ qx+9y+5z&=8 \end{aligned}$$

where $p$ and $q$ are real.

Find the values of $p$ and $q$ for which this system has:

i) a unique solution
ii) an infinite number of solutions
iii) no solution. (NICCEA)

**Answers**

**1(i) det =0**

**(ii) (4-t,t,t+1) i.e. Solution are on a straight line (spine of a book)**

**2 (i)** $\left(\frac{7t-19}{5}, \frac{37-11t}{5}, t\right)$

**(ii) Spine of a book**

**3(a) a=b=c**

**(b) (i) Equation 1 and 2 are the same and 3 is a non-parallel plane hence an infinite number of solutions**

**(ii) 2 parallel planes and the other plane intersecting both**

**4 Triangular prism**

**5(i) planes met at a point**

**(ii)Triangular prism**

**(iii) 2 parallel planes and the other plane intersecting both**

**6 Spine of a book**

**7(i)** $q \neq 2$

**(ii) q=2, p=-4**

**(iii) q=2,** $p \neq -4$

Roots of Polynomials Questions

## Exercise 4A

1 $\alpha$ and $\beta$ are the roots of the quadratic equation $3x^2 + 7x - 4 = 0$. Without solving the equation, find the values of:

**a** $\alpha + \beta$ **b** $\alpha\beta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta}$ **d** $\alpha^2 + \beta^2$

2 $\alpha$ and $\beta$ are the roots of the quadratic equation $7x^2 - 3x + 1 = 0$. Without solving the equation, find the values of:

**a** $\alpha + \beta$ **b** $\alpha\beta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta}$ **d** $\alpha^2 + \beta^2$

3 $\alpha$ and $\beta$ are the roots of the quadratic equation $6x^2 - 9x + 2 = 0$. Without solving the equation, find the values of:

**a** $\alpha + \beta$ **b** $\alpha^2 \times \beta^2$

**c** $\frac{1}{\alpha} + \frac{1}{\beta}$ **d** $\alpha^3 + \beta^3$

**Hint** Try expanding $(\alpha + \beta)^3$.

4 The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = 2$ and $\beta = -3$. Find integer values for $a$, $b$ and $c$.

5 The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{3}$ Find integer values for $a$, $b$ and $c$.

6 The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{-1 + i}{2}$ and $\beta = \frac{-1 - i}{2}$ Find integer values for $a$, $b$ and $c$.

7 One of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $\alpha = -1 - 4i$.

**a** Write down the other root, $\beta$.

**b** Given that $a = 1$, find the values of $b$ and $c$.

(P) 8 Given that $kx^2 + (k - 3)x - 2 = 0$, find the value of $k$ if the sum of the roots is 4.

(P) 9 The equation $nx^2 - (16 + n)x + 256 = 0$ has real roots $\alpha$ and $-\alpha$. Find the value of $n$.

(P) 10 The roots of the equation $6x^2 + 36x + k = 0$ are reciprocals of each other. Find the value of $k$.

(P) 11 The equation $mx^2 + 4x + 4m = 0$ has roots of the form $k$ and $2k$. Find the values of $m$ and $k$.

(P) 12 The equation $ax^2 + 8x + c = 0$, where $a$ and $c$ are real constants, has roots $\alpha$ and $\alpha^*$.

**a** Given that $\text{Re}(\alpha) = 2$, find the value of $a$.

**b** Given that $\text{Im}(\alpha) = 3i$, find the value of $c$.

(P) 13 The equation $4x^2 + px + q = 0$, where $p$ and $q$ are real constants, has roots $\alpha$ and $\alpha^*$.

**a** Given that $\text{Re}(\alpha) = -3$, find the value of $p$.

**b** Given that $\text{Im}(\alpha) \neq 0$, find the range of possible values of $q$.

ANSWERS

**Exercise 4A**

1 **a** $-\frac{7}{3}$ **b** $-\frac{4}{3}$ **c** $\frac{7}{4}$ **d** $\frac{73}{9}$

2 **a** $\frac{3}{7}$ **b** $\frac{1}{7}$ **c** 3 **d** $-\frac{5}{49}$

3 **a** $\frac{3}{2}$ **b** $\frac{1}{9}$ **c** $\frac{9}{2}$ **d** $\frac{15}{8}$

4 $a = 1, b = 1, c = -6$

5 $a = 6, b = 5, c = 1$

6 $a = 2, b = 2, c = 1$

7 **a** $-1 + 4i$ **b** $b = 2, c = 17$

8 $\frac{3}{5}$

9 $-16$

10 6

11 $k = \sqrt{2}$ and $m = -\frac{2\sqrt{2}}{3}$ or $k = -\sqrt{2}$ and $m = \frac{2\sqrt{2}}{3}$

12 **a** $-2$ **b** $-26$

13 **a** 24 **b** $q > 36$

## Exercise 4B

**1** $\alpha$, $\beta$ and $\gamma$ are the roots of the cubic equation $2x^3 + 5x^2 - 2x + 3 = 0$. Find the values of:

**a** $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \beta\gamma + \gamma\alpha$ **d** $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

**2** $\alpha$, $\beta$ and $\gamma$ are the roots of the cubic equation $x^3 + 5x^2 + 17x + 13 = 0$. Find the values of:

**a** $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \beta\gamma + \gamma\alpha$ **d** $\alpha^2\beta^2\gamma^2$

**3** $\alpha$, $\beta$ and $\gamma$ are the roots of the cubic equation $7x^3 - 4x^2 - x + 6 = 0$. Find the values of:

**a** $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha^3\beta^3\gamma^3$ **d** $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

**4** The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$ and $\gamma = 1$.
Find integer values for $a$, $b$, $c$ and $d$.

**5** The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 + 3\text{i}$, $\beta = 1 - 3\text{i}$ and $\gamma = \frac{1}{2}$
Find integer values for $a$, $b$, $c$ and $d$.

**6** The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = \frac{5}{4}$, $\beta = -\frac{3}{2}$ and $\gamma = \frac{1}{2}$
Find integer values for $a$, $b$, $c$ and $d$.

**7** The cubic equation $16x^3 - kx^2 + 1 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

**a** Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **(2 marks)**

**b** **i** Given that $\alpha = \beta$, find the roots of the equation. **(5 marks)**

**ii** Find the value of $k$. **(1 mark)**

**8** The cubic equation $2x^3 - kx^2 + 30x - 13 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

**a** Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$, and express $k$ in terms of $\alpha$, $\beta$ and $\gamma$. **(3 marks)**

**b** Given that $\alpha = 2 - 3\text{i}$, find the value of $k$. **(4 marks)**

**9** The cubic equation $x^3 - mx + n = 0$ has roots $1$, $-4$ and $\alpha$.

**a** State, with a reason, whether $\alpha$ is real. **(1 mark)**

**b** Find the values of $m$, $n$ and $\alpha$. **(4 marks)**

**10** The cubic equation $2x^3 - 10x^2 + 8x - k = 0$ has a root at $x = 3 - \text{i}$.

**a** Find the other two roots of the equation. **(4 marks)**

**b** Hence find the value of $k$. **(2 marks)**

**11** The cubic equation $x^3 - 14x^2 + 56x - 64 = 0$ has roots $\alpha$, $k\alpha$ and $k^2\alpha$ for some real constant $k$.
Find the values of $\alpha$ and $k$. **(5 marks)**

**12** Given that the roots of $8x^3 + 12x^2 - cx + d = 0$ are $\alpha$, $\dfrac{\alpha}{2}$ and $\alpha - 4$, find $\alpha$, $c$ and $d$. **(5 marks)**

**13** Given that the roots of the cubic equation $2x^3 + 48x^2 + cx + d = 0$ are $\alpha$, $2\alpha$ and $3\alpha$, find the values of $\alpha$, $c$ and $d$. **(5 marks)**

<u>ANSWERS</u>

### Exercise 4B

**1** **a** $-\frac{5}{2}$ **b** $-\frac{3}{2}$ **c** $-1$ **d** $\frac{2}{3}$

**2** **a** $-5$ **b** $-13$ **c** $17$ **d** $169$

**3** **a** $\frac{4}{7}$ **b** $-\frac{6}{7}$ **c** $-\frac{216}{343}$ **d** $\frac{1}{6}$

**4** $a = 4, b = -12, c = 11, d = -3$

**5** $a = 2, b = -5, c = 22, d = -10$

**6** $a = 16, b = -4, c = -32, d = 15$

**7** **a** $\alpha\beta + \beta\gamma + \gamma\alpha = 0$, $\alpha\beta\gamma = -\frac{1}{16}$

**b** **i** $\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \gamma = -\frac{1}{4}$ **ii** 12

**8** **a** $\alpha\beta + \beta\gamma + \gamma\alpha = 15$, $\alpha\beta\gamma = \frac{13}{2}$, $k = 2(\alpha + \beta + \gamma)$ **b** 9

**9** **a** Yes – there are two other real roots, so $\alpha^*$ couldn't also be a root.

**b** $m = 13, n = 12, \alpha = 3$

**10** **a** $-1$ and $3 + \text{i}$ **b** $-20$

**11** $\alpha = 2$ and $k = 2$ or $\alpha = 8$ and $k = \frac{1}{2}$

**12** $\alpha = 1, c = 32, d = 12$

**13** $\alpha = -4, c = 352, d = 768$

**Exercise 4C**

1 $\alpha, \beta, \gamma$ and $\delta$ are the roots of the quartic equation $4x^4 + 3x^3 + 2x^2 - 5x - 4 = 0$. Without solving the equation, find the values of:

**a** $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

**c** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **d** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

**Hint** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}$

2 $\alpha, \beta, \gamma$ and $\delta$ are the roots of the quartic equation $2x^4 + 4x^3 - 3x^2 - x + 2 = 0$. Find the values of:

**a** $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$ **c** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

**d** $\alpha\beta\gamma\delta$ **e** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

3 $\alpha, \beta, \gamma$ and $\delta$ are the roots of the quartic equation $x^4 + 3x^3 + 2x^2 - x + 4 = 0$. Find the values of:

**a** $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$ **c** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

**d** $\alpha\beta\gamma\delta$ **e** $\alpha^2\beta^2\gamma^2\delta^2$

4 $\alpha, \beta, \gamma$ and $\delta$ are the roots of the quartic equation $7x^4 + 6x^3 - 5x^2 + 4x + 3 = 0$. Find the values of:

**a** $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$ **c** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

**d** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ **e** $\alpha^3\beta^3\gamma^3\delta^3$

5 The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are $\alpha = -\frac{3}{2}$, $\beta = -\frac{1}{2}$, $\gamma = -2$ and $\delta = \frac{2}{3}$. Find integer values for $a, b, c, d$ and $e$.

6 The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are $\alpha = -\frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = 1 + \mathrm{i}$ and $\delta = 1 - \mathrm{i}$. Find integer values for $a, b, c, d$ and $e$.

7 The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are such that $\Sigma\alpha = \frac{17}{12}$, $\Sigma\alpha\beta = -\frac{25}{72}$, $\Sigma\alpha\beta\gamma = -\frac{53}{72}$ and $\alpha\beta\gamma\delta = -\frac{1}{6}$. Find integer values for $a, b, c, d$ and $e$.

8 The quartic equation $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$ has roots $\alpha$, $\alpha + k$, $\alpha + 2k$ and $\alpha + 3k$ for some real constant $k$. Solve the equation. **(7 marks)**

9 The quartic equation $3072x^4 - 2880x^3 + 840x^2 - 90x + 3 = 0$ has roots $\alpha$, $r\alpha$, $r^2\alpha$ and $r^3\alpha$ for some real constant $r$. Solve the equation. **(7 marks)**

10 Three of the roots of the quartic equation $40x^4 + 90x^3 - 115x^2 + mx + n = 0$ are 1, $-3$ and $\frac{1}{2}$

**a** Find the fourth root. **(2 marks)**

**b** Find the values of $m$ and $n$. **(4 marks)**

11 The quartic equation $2x^4 - 34x^3 + 202x^2 + dx + e = 0$ has roots $\alpha$, $\alpha + 1$, $2\alpha + 1$ and $3\alpha + 1$.

**a** Find $\alpha$. **(2 marks)**

**b** Find the values of $d$ and $e$. **(4 marks)**

12 The equation $4x^4 - 19x^3 + px^2 + qx + 10 = 0$, $x \in \mathbb{C}$, $p, q \in \mathbb{R}$, has roots $\alpha, \beta, \gamma$ and $\delta$. Given that $\gamma = 3 + \mathrm{i}$ and $\delta = \gamma^*$,

**a** show that $4\alpha + 4\beta + 5 = 0$ and that $4\alpha\beta - 1 = 0$. **(2 marks)**

**b** Hence find all the roots of the quartic equation and find the values of $p$ and $q$. **(5 marks)**

**c** Show these roots on an Argand diagram. **(3 marks)**

13 A quartic equation $6x^4 - 10x^3 + 3x^2 + 6x - 40 = 0$ has roots $\alpha, \beta, \gamma$ and $\delta$.

**a** Show that $\frac{1 - 3\mathrm{i}}{2}$ is one root of the equation. **(3 marks)**

**b** Without solving the equation, find the other roots. **(5 marks)**

**c** Show these roots on an Argand diagram. **(3 marks)**

## ANSWERS

**Exercise 4C**

1 **a** $-\frac{3}{4}$ **b** $\frac{1}{2}$ **c** $\frac{5}{4}$ **d** $\frac{5}{4}$

2 **a** $-2$ **b** $-\frac{3}{2}$ **c** $\frac{1}{2}$ **d** 1 **e** $\frac{1}{2}$

3 **a** $-3$ **b** 2 **c** 1 **d** $\frac{1}{4}$ **e** 16

4 **a** $-\frac{6}{7}$ **b** $-\frac{5}{7}$ **c** $-\frac{4}{7}$ **d** $-\frac{4}{3}$ **e** $\frac{27}{343}$

5 $a = 12, b = 40, c = 25, d = -20, e = -12$

6 $a = 6, b= -11, c = 9, d = 4, e = -2$

7 $a = 72, b = -102, c = -25, d = 53, e = -12$

8 $x = 1, 3, 5$ or $7$

9 $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ or $\frac{1}{16}$

10 **a** $-\frac{3}{4}$ **b** $m = -60, n = 45$

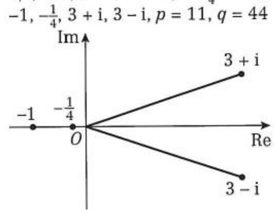
11 **a** 2 **b** $d = -494, e = 420$

12 **a** $\alpha + \beta + (3 + \mathrm{i}) + (3 - \mathrm{i}) = \frac{19}{4} \Rightarrow 4\alpha + 4\beta + 5 = 0$

$\alpha\beta(3 + \mathrm{i})(3 - \mathrm{i}) = 10\alpha\beta = \frac{10}{4} \Rightarrow 4\alpha\beta - 1 = 0$

**b** $-1, -\frac{1}{4}, 3 + \mathrm{i}, 3 - \mathrm{i}, p = 11, q = 44$

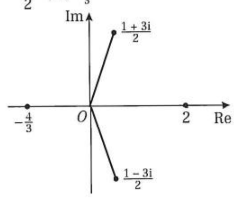
**c**




13 **a** $6\left(\frac{1 - 3\mathrm{i}}{2}\right)^4 - 10\left(\frac{1 - 3\mathrm{i}}{2}\right)^3 + 3\left(\frac{1 - 3\mathrm{i}}{2}\right)^2 + 6\left(\frac{1 - 3\mathrm{i}}{2}\right) - 40 = 0$

**b** $\frac{1 + 3\mathrm{i}}{2}, 2, -\frac{4}{3}$

**c**



## Exercise 4D

1 A quadratic equation has roots $\alpha$ and $\beta$. Given that $\alpha + \beta = 4$ and $\alpha\beta = 3$, find:

a $\frac{1}{\alpha} + \frac{1}{\beta}$ b $\alpha^2\beta^2$ c $\alpha^2 + \beta^2$ d $\alpha^3 + \beta^3$

2 A quadratic equation has roots $\alpha$ and $\beta$. Given that $\alpha + \beta = -\frac{2}{3}$ and $\alpha\beta = \frac{3}{4}$, find:

a $\frac{1}{\alpha} + \frac{1}{\beta}$ b $\alpha^2\beta^2$ c $\alpha^2 + \beta^2$ d $\alpha^3 + \beta^3$

3 A quadratic equation has roots $\alpha$ and $\beta$. Given that $\alpha + \beta = \frac{5}{4}$ and $\alpha\beta = -\frac{1}{3}$, find:

a $(\alpha + 2)(\beta + 2)$ b $(\alpha - 4)(\beta - 4)$ c $(\alpha^2 + 1)(\beta^2 + 1)$

4 A cubic equation has roots $\alpha$, $\beta$ and $\gamma$. Given that $\alpha + \beta + \gamma = 2$, $\alpha\beta + \beta\gamma + \gamma\alpha = -3$ and $\alpha\beta\gamma = 4$, find:

a $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ b $\alpha^2 + \beta^2 + \gamma^2$ c $\alpha^3 + \beta^3 + \gamma^3$ d $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$

5 A cubic equation has roots $\alpha$, $\beta$ and $\gamma$. Given that $\Sigma\alpha = \frac{3}{2}$, $\Sigma\alpha\beta = -\frac{4}{3}$ and $\alpha\beta\gamma = \frac{1}{2}$, find:

a $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ b $\alpha^2 + \beta^2 + \gamma^2$ c $\alpha^3 + \beta^3 + \gamma^3$ d $\alpha^3\beta^3\gamma^3$

6 A cubic equation has roots $\alpha$, $\beta$ and $\gamma$. Given that $\alpha + \beta + \gamma = -\frac{1}{2}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{3}{4}$ and $\alpha\beta\gamma = -\frac{2}{5}$, find:

a $(\alpha + 2)(\beta + 2)(\gamma + 2)$ b $(\alpha - 3)(\beta - 3)(\gamma - 3)$ c $(1 - \alpha)(1 - \beta)(1 - \gamma)$

d $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$ e $(\alpha\beta)^3 + (\beta\gamma)^3 + (\gamma\alpha)^3$

7 A quartic equation has roots $\alpha$, $\beta$, $\gamma$ and $\delta$. Given that $\alpha + \beta + \gamma + \delta = 3$, $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 5$, $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -4$ and $\alpha\beta\gamma\delta = -2$, find:

a $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ b $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ c $\alpha^4\beta^4\gamma^4\delta^4$

8 A quartic equation has roots $\alpha$, $\beta$, $\gamma$ and $\delta$. Given that $\Sigma\alpha = \frac{1}{2}$, $\Sigma\alpha\beta = -\frac{3}{4}$, $\Sigma\alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find:

a $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ b $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ c $\alpha^3\beta^3\gamma^3\delta^3$

d $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 + (\gamma\delta)^2 + (\alpha\delta)^2 + (\beta\delta)^2$

e $(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2$

9 A quartic equation has roots $\alpha$, $\beta$, $\gamma$ and $\delta$. Given that $\Sigma\alpha = -\frac{1}{2}$, $\Sigma\alpha\beta = -\frac{1}{3}$, $\Sigma\alpha\beta\gamma = \frac{1}{4}$ and $\alpha\beta\gamma\delta = \frac{3}{2}$, find:

a $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$ b $(2 - \alpha)(2 - \beta)(2 - \gamma)(2 - \delta)$

10 The roots of the equation $x^3 - 6x^2 + 9x - 15 = 0$ are $\alpha$, $\beta$ and $\gamma$.

a Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **(1 mark)**

b Hence find the values of:

i $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **(2 marks)**

ii $\alpha^2 + \beta^2 + \gamma^2$ **(2 marks)**

iii $(\alpha - 1)(\beta - 1)(\gamma - 1)$ **(3 marks)**

11 The roots of the equation $2x^3 + 4x^2 + 7 = 0$ are $\alpha$, $\beta$ and $\gamma$.

a Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **(1 mark)**

b Hence find the values of:

i $\alpha^2 + \beta^2 + \gamma^2$ **(2 marks)**

ii $\alpha^3\beta^3\gamma^3$ **(2 marks)**

iii $(\alpha + 2)(\beta + 2)(\gamma + 2)$ **(3 marks)**

**12** Show that $\alpha^3 + \beta^3 + \gamma^3 \equiv (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$.

**13** The roots of the equation $3x^3 - px + 11 = 0$ are $\alpha$, $\beta$ and $\gamma$.

**a** Given that $\alpha\beta + \beta\gamma + \gamma\alpha = 4$, write down the value of $p$. **(1 mark)**

**b** Write down the values of $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$. **(1 mark)**

**c** Hence find the value of $(3 - \alpha)(3 - \beta)(3 - \gamma)$. **(3 marks)**

**14** The roots of the equation $x^4 + 2x^2 - x + 3 = 0$ are $\alpha$, $\beta$, $\gamma$ and $\delta$.

**a** Write down the values of $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$ and $\alpha\beta\gamma\delta$. **(1 mark)**

**b** Hence find the values of:

**i** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ **(3 marks)**

**ii** $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ **(3 marks)**

**iii** $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$ **(3 marks)**

**15** The roots of the equation $ax^4 + 3x^3 + 2x^2 + x - 6 = 0$ are $\alpha$, $\beta$, $\gamma$ and $\delta$.

**a** Given that $\alpha\beta\gamma\delta = -3$, write down the value of $a$. **(1 mark)**

**b** Write down the values of $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\Sigma\alpha\beta\gamma$. **(1 mark)**

**c** Hence find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ **(3 marks)**

**16** Prove that if a quartic equation has roots $\alpha$, $\beta$, $\gamma$ and $\delta$ then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 \equiv (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$.

ANSWERS

## Exercise 4D

**1** **a** $\frac{4}{3}$ **b** 9 **c** 10 **d** 28

**2** **a** $-\frac{8}{9}$ **b** $\frac{9}{16}$ **c** $-\frac{19}{18}$ **d** $\frac{65}{54}$

**3** **a** $\frac{37}{6}$ **b** $\frac{32}{3}$ **c** $\frac{481}{144}$

**4** **a** $-\frac{3}{4}$ **b** 10 **c** 38 **d** $-7$

**5** **a** $-\frac{8}{3}$ **b** $\frac{59}{12}$ **c** $\frac{87}{8}$ **d** $\frac{1}{8}$

**6** **a** $\frac{71}{10}$ **b** $-\frac{683}{20}$ **c** $\frac{53}{20}$ **d** $\frac{13}{80}$ **e** $\frac{723}{1600}$

**7** **a** 2 **b** $-1$ **c** 16

**8** **a** $-\frac{3}{20}$ **b** $\frac{7}{4}$ **c** $\frac{64}{27}$ **d** $\frac{823}{240}$ **e** $\frac{51}{25}$

**9** **a** $\frac{23}{12}$ **b** $\frac{59}{3}$

**10** **a** $\alpha + \beta + \gamma = 6$, $\alpha\beta + \beta\gamma + \gamma\alpha = 9$, $\alpha\beta\gamma = 15$

**b** **i** $\frac{3}{5}$ **ii** 18 **iii** 11

**11** **a** $\alpha + \beta + \gamma = -2$, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$, $\alpha\beta\gamma = -\frac{7}{2}$

**b** **i** 4 **ii** $-\frac{343}{8}$ **iii** $-\frac{7}{2}$

**12**
$$\begin{aligned}(\alpha + \beta + \gamma)^3 &\equiv (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha))\\ &\equiv \alpha^3 + \beta^3 + \gamma^3 + \alpha(\beta^2 + \gamma^2) + \beta(\alpha^2 + \gamma^2)\\ &\quad + \gamma(\alpha^2 + \beta^2) + 2(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)\end{aligned}$$
$$\begin{aligned}(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)\\ &\equiv \alpha^2\beta + \beta^2\alpha + \alpha^2\gamma + \gamma^2\alpha + \beta^2\gamma + \gamma^2\beta + 3\alpha\beta\gamma\\ &\equiv \alpha(\beta^2 + \gamma^2) + \beta(\alpha^2 + \gamma^2) + \gamma(\alpha^2 + \beta^2) + 3\alpha\beta\gamma\end{aligned}$$
$$(\alpha + \beta + \gamma)^3 \equiv \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma$$
$$\alpha^3 + \beta^3 + \gamma^3 \equiv (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

**13** **a** $-12$ **b** $\alpha + \beta + \gamma = 0$, $\alpha\beta\gamma = -\frac{11}{3}$ **c** $\frac{128}{3}$

**14** **a** $\sum\alpha = 0$, $\sum\alpha\beta = 2$, $\sum\alpha\beta\gamma = 1$, $\alpha\beta\gamma\delta = 3$

**b** **i** $\frac{1}{3}$ **ii** $-4$ **iii** 7

**15** **a** 2 **b** $\sum\alpha = -\frac{3}{2}$, $\sum\alpha\beta = 1$, $\sum\alpha\beta\gamma = -\frac{1}{2}$ **c** $\frac{1}{6}$

**16**
$$\begin{aligned}\left(\sum\alpha\right)^2 &\equiv (\alpha + \beta + \gamma + \delta)^2\\ &\equiv \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \beta\gamma + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta)\end{aligned}$$
$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \equiv \left(\sum\alpha\right)^2 - 2(\alpha\beta + \beta\gamma + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta)$$

## Exercise 4E

1 The cubic equation $x^3 - 7x^2 + 6x + 5 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.
Find equations with roots:

a $(\alpha + 1)$, $(\beta + 1)$ and $(\gamma + 1)$

b $2\alpha$, $2\beta$ and $2\gamma$

2 The cubic equation $3x^3 - 4x^2 - 5x + 1 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.
Find equations with roots:

a $(\alpha - 3)$, $(\beta - 3)$ and $(\gamma - 3)$

b $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$

3 The cubic equation $x^3 - 3x^2 + 4x - 7 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.
Without solving the equation, find the equation with roots $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$.
Give your answer in the form $aw^3 + bw^2 + cw + d = 0$ where $a$, $b$, $c$ and $d$ are integers to be determined. **(5 marks)**

4 The cubic equation $x^3 + 4x^2 - 4x + 2 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.
Without solving the equation, find the equation with roots $(2\alpha - 1)$, $(2\beta - 1)$ and $(2\gamma - 1)$.
Give your answer in the form $w^3 + pw^2 + qw + r = 0$ where $p$, $q$ and $r$ are integers to be found. **(5 marks)**

5 The cubic equation $3x^3 - x^2 + 2x - 5 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.
Without solving the equation, find the equation with roots $(3\alpha + 1)$, $(3\beta + 1)$ and $(3\gamma + 1)$.
Give your answer in the form $aw^3 + bw^2 + cw + d = 0$ where $a$, $b$, $c$ and $d$ are integers to be determined. **(5 marks)**

6 The quartic equation $2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$ has roots $\alpha$, $\beta$, $\gamma$ and $\delta$. Find equations with integer coefficients that have roots:

a $3\alpha$, $3\beta$, $3\gamma$ and $3\delta$ b $(\alpha - 1)$, $(\beta - 1)$, $(\gamma - 1)$ and $(\delta - 1)$

7 The quartic equation $x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$ has roots $\alpha$, $\beta$, $\gamma$ and $\delta$.
Without solving the equation, find equations with integer coefficients that have roots:

a $2\alpha$, $2\beta$, $2\gamma$ and $2\delta$ **(6 marks)**

b $(\alpha - 2)$, $(\beta - 2)$, $(\gamma - 2)$ and $(\delta - 2)$ **(6 marks)**

8 The quartic equation $3x^4 + 5x^3 - 4x^2 - 3x + 1 = 0$ has roots $\alpha$, $\beta$, $\gamma$ and $\delta$.
Without solving the equation, find equations with integer coefficients that have roots:

a $3\alpha$, $3\beta$, $3\gamma$ and $3\delta$ **(6 marks)**

b $(\alpha + 1)$, $(\beta + 1)$, $(\gamma + 1)$ and $(\delta + 1)$ **(6 marks)**

ANSWERS

**Exercise 4E**

1 a $w^3 - 10w^2 + 23w - 9 = 0$
b $w^3 - 14w^2 + 24w + 40 = 0$

2 a $3w^3 + 23w^2 + 52w + 31 = 0$
b $24w^3 - 16w^2 - 10w + 1 = 0$

3 $w^3 - 9w^2 + 31w - 79 = 0$

4 $w^3 + 11w^2 + 3w + 9 = 0$

5 $w^3 - 4w^2 + 11w - 53 = 0$

6 a $2w^4 + 12w^3 - 45w^2 + 54w - 81 = 0$
b $2w^4 + 12w^3 + 19w^2 + 12w + 2=0$

7 a $w^4 + 4w^3 - 12w^2 + 32w + 80 = 0$
b $w^4 + 10w^3 + 33w^2 + 48w + 33 = 0$

8 a $w^4 + 5w^3 - 12w^2 - 27w + 27 = 0$
b $3w^4 - 7w^3 - w^2 + 8w - 2 = 0$

## Mixed exercise 4

1 The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are $\alpha = \frac{1}{5}$, $\beta = -\frac{2}{5}$, $\gamma = -\frac{3}{5}$ and $\delta = -\frac{1}{2}$
Find integer values for $a$, $b$, $c$, $d$ and $e$.

2 The cubic equation $x^3 + px^2 + 37x - 52 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

**a** Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$, and express $p$ in terms of $\alpha$, $\beta$ and $\gamma$. **(3 marks)**

**b** Given that $\alpha = 3 - 2i$, find the value of $p$. **(4 marks)**

3 The cubic equation $2x^3 + 5x^2 - 2x + q = 0$ has a root at $x = -2 + i$.

**a** Find the other two roots of the equation. **(4 marks)**

**b** Hence find the value of $q$. **(2 marks)**

4 The quartic equation $x^4 - 40x^3 + 510x^2 - 2200x + 1729 = 0$ has roots $\alpha$, $\alpha + 2k$, $\alpha + 4k$ and $\alpha + 6k$ for some real constant $k$. Solve the equation. **(7 marks)**

5 Three of the roots of the quartic equation $24x^4 - 58x^3 + 17x^2 + dx + e = 0$ are $\frac{1}{2}$, $-\frac{1}{3}$ and 2.

**a** Find the fourth root. **(2 marks)**

**b** Find the values of $d$ and $e$. **(4 marks)**

6 The equation $x^4 + 2x^3 + mx^2 + nx + 85 = 0$, $x \in \mathbb{C}$, $m, n \in \mathbb{R}$, has roots $\alpha$, $\beta$, $\gamma$ and $\delta$.
Given that $\alpha = -2 + i$ and $\beta = \alpha^*$,

**a** show that $\gamma + \delta - 2 = 0$ and that $\gamma\delta - 17 = 0$. **(2 marks)**

**b** Hence find all the roots of the quartic equation and find the values of $m$ and $n$. **(5 marks)**

**c** Show these roots on an Argand diagram. **(3 marks)**

7 A quartic equation $4x^4 - 16x^3 + 115x^2 + 4x - 29 = 0$ has roots $\alpha$, $\beta$, $\gamma$ and $\delta$.

**a** Show that $2 - 5i$ is one root of the equation. **(3 marks)**

**b** Without solving the equation, find the other roots. **(5 marks)**

**c** Show these roots on an Argand diagram. **(3 marks)**

8 The roots of the equation $2x^3 - 5x^2 + 11x - 9 = 0$ are $\alpha$, $\beta$ and $\gamma$.

**a** Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **(1 mark)**

**b** Hence find the values of:

**i** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **(2 marks)**

**ii** $\alpha^2 + \beta^2 + \gamma^2$ **(2 marks)**

**iii** $(\alpha - 1)(\beta - 1)(\gamma - 1)$ **(3 marks)**

9 The roots of the equation $px^4 + 12x^3 + 6x^2 + 5x - 7 = 0$ are $\alpha$, $\beta$, $\gamma$ and $\delta$.

**a** Given that $\alpha\beta\gamma\delta = -1$, write down the value of $p$. **(1 mark)**

**b** Write down the values of $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\Sigma\alpha\beta\gamma$. **(1 mark)**

**c** Hence find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. **(3 marks)**

10 The roots of the equation $5x^3 + cx + 21 = 0$ are $\alpha$, $\beta$ and $\gamma$.

**a** Given that $\alpha\beta + \beta\gamma + \gamma\alpha = -6$, write down the value of $c$. **(1 mark)**

**b** Write down values for $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$. **(1 mark)**

**c** Hence find the value of $(1 - \alpha)(1 - \beta)(1 - \gamma)$. **(3 marks)**

11 The cubic equation $2x^3 + 5x^2 + 7x - 2 = 0$ has roots $\alpha$, $\beta$ and $\gamma$.
Without solving the equation, find the equation with roots $(3\alpha + 1)$, $(3\beta + 1)$ and $(3\gamma + 1)$.
Give your answer in the form $pw^3 + qw^2 + rw + s = 0$ where $p$, $q$, $r$ and $s$ are integers to be found. **(5 marks)**

12 The quartic equation $6x^4 - 2x^3 - 5x^2 + 7x + 8 = 0$ has roots $\alpha$, $\beta$, $\gamma$ and $\delta$.
Without solving the equation, find equations with integer coefficients that have roots:

**a** $2\alpha$, $2\beta$, $2\gamma$ and $2\delta$ **(6 marks)**

**b** $(3\alpha - 2)$, $(3\beta - 2)$, $(3\gamma - 2)$ and $(3\delta - 2)$ **(6 marks)**

ANSWERS

## Mixed exercise 4

1 $a = 250$, $b = 325$, $c = 110$, $d = -7$, $e = -6$

2 a $\alpha\beta + \beta\gamma + \gamma\alpha = 37$, $\alpha\beta\gamma = 52$, $p = -\alpha - \beta - \gamma$

b $-10$

3 a $-2 - \mathrm{i}, \frac{3}{2}$  b $-15$

4 $x = 1, 7, 13$ or $19$

5 a $\frac{1}{4}$

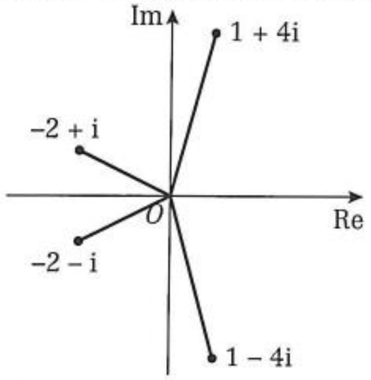
b $d = 7$, $e = -2$

6 a $(-2 + \mathrm{i}) + (-2 - \mathrm{i}) + \gamma + \delta = -2 \Rightarrow \gamma + \delta - 2 = 0$

$(-2 + \mathrm{i})(-2 - \mathrm{i})\gamma\delta = 85 \Rightarrow 5\gamma\delta = 85 \Rightarrow \gamma\delta - 17 = 0$

b Roots: $-2 \pm \mathrm{i}$, $1 \pm 4\mathrm{i}$; $m = 14$, $n = 58$

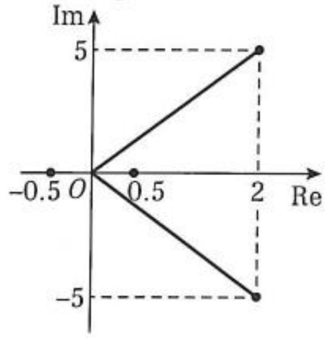
c




7 a $4(2 - 5\mathrm{i})^4 - 16(2 - 5\mathrm{i})^3 + 115(2 - 5\mathrm{i})^2 + 4(2 - 5\mathrm{i}) - 29 = 0$

b $2 + 5\mathrm{i}, \pm\frac{1}{2}$

c




8 a $\alpha + \beta + \gamma = \frac{5}{2}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{11}{2}$, $\alpha\beta\gamma = \frac{9}{2}$

b i $\frac{11}{9}$  ii $-\frac{19}{4}$  iii $\frac{1}{2}$

9 a $7$  b $\sum\alpha = -\frac{12}{7}$, $\sum\alpha\beta = \frac{6}{7}$, $\sum\alpha\beta\gamma = -\frac{5}{7}$  c $\frac{60}{49}$

10 a $-30$  b $\alpha + \beta + \gamma = 0$, $\alpha\beta\gamma = -\frac{21}{5}$  c $-\frac{4}{5}$

11 $2w^3 + 9w^2 + 39w - 104 = 0$

12 a $3w^4 - 2w^3 - 10w^2 + 28w + 64 = 0$

b $2w^4 + 14w^3 + 21w^2 + 43w + 298 = 0$

Vectors Questions

*UPM Ex2B Q3,4,6,7,9,10,11,13,18,19,20

3. If $a\mathbf{i} + 8\mathbf{j}$ is parallel to $2\mathbf{i} + 4\mathbf{j}$, find the value of $\mathbf{a}$.
4. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ find $\hat{\mathbf{a}}$, a unit vector in the direction of $\mathbf{a}$.
5. If $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$ find $\hat{\mathbf{b}}$, a unit vector in the direction of $\mathbf{b}$.
6. Find a vector that is of magnitude 39 units and is parallel to $5\mathbf{i} + 12\mathbf{j}$.
7. Find a vector that is of magnitude $3\sqrt{5}$ units and is parallel to $2\mathbf{i} - \mathbf{j}$.
8. Find a vector that is of magnitude 2 units and is parallel to $4\mathbf{i} - 3\mathbf{j}$.
9. If the point P has position vector $2\mathbf{i} + 3\mathbf{j}$ and point Q has position vector $7\mathbf{i} + 4\mathbf{j}$, find: (a) $\overrightarrow{PQ}$, (b) $\overrightarrow{QP}$.
10. If the point P has position vector $7\mathbf{i} - 3\mathbf{j}$ and point Q has position vector $5\mathbf{i} + 5\mathbf{j}$, find: (a) $\overrightarrow{PQ}$, (b) $\overrightarrow{QP}$.

11. The point P has position vector $-5\mathbf{i} + 3\mathbf{j}$ and Q is a point such that $\overrightarrow{PQ} = 7\mathbf{i} - \mathbf{j}$. Find the position vector of Q.
12. The point P has position vector $3\mathbf{i} - 2\mathbf{j}$ and Q is a point such that $\overrightarrow{QP} = 2\mathbf{i} - 3\mathbf{j}$. Find the position vector of Q.
13. Using $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ as base vectors, express the vectors $\mathbf{c} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ in the form $\lambda\mathbf{a} + \mu\mathbf{b}$.

18. The point K has position vector $3\mathbf{i} + 2\mathbf{j}$ and point L has position vector $\mathbf{i} + 3\mathbf{j}$. Find the position vector of the point which divides KL in the ratio (a) $4:3$, (b) $4:-3$.
19. The three points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. If $\mathbf{c} = 3\mathbf{b} - 2\mathbf{a}$, show that A, B and C are collinear.
20. The three points A, B and C have position vectors $\mathbf{i} - \mathbf{j}$, $5\mathbf{i} - 3\mathbf{j}$ and $11\mathbf{i} - 6\mathbf{j}$ respectively. Show that A, B and C are collinear.

ANSWERS Ex2B

1. $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OB} = 3\mathbf{i}$, $\overrightarrow{OC} = -\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OD} = -3\mathbf{i} + 2\mathbf{j}$, $\overrightarrow{OE} = -2\mathbf{i} - \mathbf{j}$, $\overrightarrow{OF} = -2\mathbf{i} - 2\mathbf{j}$, $\overrightarrow{OG} = -2\mathbf{j}$, $\overrightarrow{OH} = 3\mathbf{i} - 3\mathbf{j}$
2. (a) (i) $4\mathbf{i} + 3\mathbf{j}$ (ii) 5 units (iii) $36{\cdot}8°$ (b) (i) $-4\mathbf{i} + \mathbf{j}$ (ii) $4{\cdot}12$ units (iii) $14°$ (c) (i) $2\mathbf{i} - 3\mathbf{j}$ (ii) $3{\cdot}61$ units (iii) $56{\cdot}3°$

3. 4 4. $\frac{\sqrt{5}}{5}(\mathbf{i} + 2\mathbf{j})$ 5. $\frac{\sqrt{10}}{10}(3\mathbf{i} - \mathbf{j})$ 6. $15\mathbf{i} + 36\mathbf{j}$ 7. $6\mathbf{i} - 3\mathbf{j}$

8. $\frac{2}{5}(4\mathbf{i} - 3\mathbf{j})$ 9. (a) $5\mathbf{i} + \mathbf{j}$ (b) $-5\mathbf{i} - \mathbf{j}$ 10. (a) $-2\mathbf{i} + 8\mathbf{j}$ (b) $2\mathbf{i} - 8\mathbf{j}$
11. $2\mathbf{i} + 2\mathbf{j}$ 12. $\mathbf{i} + \mathbf{j}$
13. $\mathbf{a} + 2\mathbf{b}$, $\frac{3}{11}\mathbf{a} + \frac{23}{11}\mathbf{b}$ 14. (a) $\sqrt{10}$ (b) $\sqrt{5}$ (c) $5\mathbf{i}$ (d) 5
15. (a) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (b) 5 (c) $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$ (d) $\sqrt{58}$ 16. $6\mathbf{i} + \mathbf{j}$ 17. $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$
18. (a) $1\frac{6}{7}\mathbf{i} + 2\frac{4}{7}\mathbf{j}$ (b) $-5\mathbf{i} + 6\mathbf{j}$ 21. C, E and F

*UPM Ex2C Q1,3,5,6,8,11,12,15,16

1. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$, $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{d} = 6\mathbf{i} + 3\mathbf{j}$, find which two of these vectors are perpendicular to each other.
2. If $\mathbf{e} = -\mathbf{i} - 3\mathbf{j}$, $\mathbf{f} = \mathbf{i} + 3\mathbf{j}$, $\mathbf{g} = -3\mathbf{i} - 2\mathbf{j}$ and $\mathbf{h} = 6\mathbf{i} - 9\mathbf{j}$, find which two of these vectors are perpendicular to each other.
3. Find the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ given that $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{b} = 5\mathbf{i} + 12\mathbf{j}$. (Give your answer to the nearest degree.)
4. Find the angle between the vectors $\mathbf{c}$ and $\mathbf{d}$ given that $\mathbf{c} = 5\mathbf{i} - \mathbf{j}$ and $\mathbf{d} = 2\mathbf{i} + 3\mathbf{j}$. (Give your answer to the nearest degree.)
5. Find the angle between the vectors $\mathbf{e}$ and $\mathbf{f}$ given that $\mathbf{e} = -\mathbf{i} - 2\mathbf{j}$ and $\mathbf{f} = 2\mathbf{i} + \mathbf{j}$. (Give your answer to the nearest degree.)
6. If the angle between the vectors $\mathbf{c} = a\mathbf{i} + 2\mathbf{j}$ and $\mathbf{d} = 3\mathbf{i} + \mathbf{j}$ is 45° find the two possible values of $a$.

8. The points A, B, C and D have position vectors $5\mathbf{i} + \mathbf{j}$, $-3\mathbf{i} + 2\mathbf{j}$, $-3\mathbf{i} - 3\mathbf{j}$ and $\mathbf{i} - 6\mathbf{j}$ respectively. Show that AC is perpendicular to BD.
9. The points E, F and G have position vectors $2\mathbf{i} + 2\mathbf{j}$, $\mathbf{i} + 6\mathbf{j}$ and $-7\mathbf{i} + 4\mathbf{j}$. Show that the triangle EFG is right-angled at F.
10. The points A, B, C and D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ respectively where $\mathbf{a} = -2\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 4\mathbf{j}$, $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{d} = 4\mathbf{i} + y\mathbf{j}$. If AC is perpendicular to BD, find the value of $y$.
11. The points A, B, C and D have position vectors $-2\mathbf{i} + \mathbf{j}$, $7\mathbf{j}$, $3\mathbf{i} + 6\mathbf{j}$ and $x\mathbf{i} + y\mathbf{j}$ respectively. If $|\overrightarrow{AC}| = |\overrightarrow{BD}|$ and AC is perpendicular to BD, find the two possible values of $x$ and the corresponding values of $y$.

12. If $\mathbf{a} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ find:
   (a) a unit vector parallel to $\mathbf{a}$,
   (b) a unit vector perpendicular to $\mathbf{a}$.

15. If $\overrightarrow{OA} = 2\mathbf{a} + 3\mathbf{b}$ and $\overrightarrow{OB} = 3\mathbf{a} - 2\mathbf{b}$ show that $\overrightarrow{OA} \cdot \overrightarrow{OB} = 6a^2 + 5\mathbf{a} \cdot \mathbf{b} - 6b^2$.
16. If $\overrightarrow{OC} = 2\mathbf{a} + 3\mathbf{b}$ and $\overrightarrow{OD} = 2\mathbf{a} - 3\mathbf{b}$ show that $\overrightarrow{OC} \cdot \overrightarrow{OD} = 4a^2 - 9b^2$.

ANSWERS Ex2C

**1.** b and d **2.** g and h **3.** 14° **4.** 68° **5.** 143° **6.** −4 or 1
**7.** (a) $x^2 + y^2 = 25$ (b) $2x + y = 0$ (c) $x = 2y$ (d) $4x + y = 9$ **10.** 1
**11.** $x = 5, y = 2; x = -5, y = 12$ **12.** (a) $\begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ (b) $\pm\begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$ **13.** (a) $\begin{pmatrix} 7\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$ (b) $\pm\begin{pmatrix} 3 \\ 21 \end{pmatrix}$
**14.** (a) $-3\mathbf{i} + 4\mathbf{j}$ (b) $-6\mathbf{i} - \mathbf{j}$ (c) $9\mathbf{i} - 3\mathbf{j}$ (d) 35°, 117°, 28° **17.** (b) 23 **18.** 131°
**19.** 4 **20.** $\frac{1}{13}\sqrt{13}$ **21.** $\frac{9}{5}\sqrt{5}$ **22.** (b) $k\sqrt{7}, 2k\sqrt{21}$

*UPM Ex17A Q1,2,4,6,7,11,12

**1.** If $\mathbf{a} = 9\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ find
(a) $|\mathbf{a}|$ (b) $|\mathbf{b}|$ (c) $|\mathbf{c}|$ (d) $\mathbf{a} \cdot \mathbf{b}$ (e) $\mathbf{b} \cdot \mathbf{c}$
(f) the angle between $\mathbf{a}$ and $\mathbf{b}$ (to the nearest degree)
(g) the angle between $\mathbf{b}$ and $\mathbf{c}$ (to the nearest degree).

**2.** If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ find

(a) $|\mathbf{a}|$ (b) $|\mathbf{b}|$ (c) the angle between $\mathbf{a}$ and $\mathbf{b}$ (to the nearest degree).

**3.** If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ find $\hat{\mathbf{a}}$, a unit vector in the direction of $\mathbf{a}$.

**4.** If $\mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$ find $\hat{\mathbf{b}}$, a unit vector in the direction of $\mathbf{b}$.

**5.** Find a vector that is perpendicular to $5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

**6.** Find a unit vector that is perpendicular to $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

**7.** State which of the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ or $\mathbf{d}$ listed below are perpendicular to the vector $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.
$\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ $\quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
$\mathbf{c} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ $\quad \mathbf{d} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

**11.** Points A, B and C have position vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ and $5\mathbf{i} + 12\mathbf{j} - 7\mathbf{k}$ respectively. Prove that A, B and C are collinear.

**12.** Points A, B and C have position vectors $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} - \mathbf{k}$ respectively. Prove that angle $B\hat{A}C$ is a right angle.

ANSWERS Ex17A

**1.** (a) 11 (b) 7 (c) 3 (d) 12 (e) 16 (f) 81° (g) 40°

**2.** (a) $\sqrt{11}$ (b) $\sqrt{6}$ (c) 76° **3.** $\frac{3}{13}\mathbf{i} + \frac{4}{13}\mathbf{j} + \frac{12}{13}\mathbf{k}$ **4.** $\begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{7}{9} \end{pmatrix}$

**5.** $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is one example **6.** $\frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ is one example

**7.** $\mathbf{c}$ and $\mathbf{d}$ **8.** $14 : -2 : 5$, $\frac{14}{15}, -\frac{2}{15}, \frac{1}{3}$ **9.** $1 : 2 : -2$
**10.** (a) $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ or any multiple thereof (b) $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$ (c) $\frac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$
**15.** 36°, 68°, 76° **17.** $4\mathbf{a} + \mathbf{b} - 2\mathbf{c}$ **18.** $3\mathbf{a} - \mathbf{b} + 2\mathbf{c}$

*UPM Ex17C Q1-4,6,7,9a,10-13,15-17,19,20ac

1. State the vector equation of the line which is parallel to $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and which passes through the point A, position vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
2. State the vector equation of the line which passes through the point B, position vector $\begin{pmatrix}-1\\2\\1\end{pmatrix}$ and which is parallel to the vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$.
3. State a vector that is parallel to the line with vector equation $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$.
4. Show that the point with position vector $4\mathbf{i} - \mathbf{j} + 12\mathbf{k}$ lies on the line with vector equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$.
5. Points A, B and C have position vectors $\begin{pmatrix}5\\2\\3\end{pmatrix}$, $\begin{pmatrix}-4\\5\\-1\end{pmatrix}$ and $\begin{pmatrix}8\\1\\7\end{pmatrix}$ respectively. Find which of these points lie on the line with vector equation $\mathbf{r} = \begin{pmatrix}-1\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$.
6. Points D, E and F have position vectors $\mathbf{i} - 2\mathbf{j}$, $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $7\mathbf{i} - 8\mathbf{j} - 4\mathbf{k}$ respectively. Find which of these points lie on the line with vector equation $\mathbf{r} = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} - \mathbf{k})$.
7. If the point A, position vector $a\mathbf{i} + b\mathbf{j} + 3\mathbf{k}$, lies on the line L, vector equation $\mathbf{r} = (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$, find the values of $a$ and $b$.

9. Find the cartesian equations of the lines with vector equations
   (a) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$,
   (b) $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$,
   (c) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \eta(2\mathbf{i} - \mathbf{j} - \mathbf{k})$.
10. Find the vector equation of the line with parametric equations $x = 2 + 3\lambda$, $y = 5 - 2\lambda$, $z = 4 - \lambda$
11. Find the vector equations of the lines with the following cartesian equations
    (a) $\dfrac{x-2}{3} = \dfrac{y-2}{2} = \dfrac{z+1}{4}$ (b) $x - 3 = \dfrac{y+2}{4} = \dfrac{z-3}{-1}$
12. Lines $L_1$ and $L_2$ have vector equations $\mathbf{r} = 8\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(-4\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = -2\mathbf{i} + 8\mathbf{j} - \mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ respectively. Show that $L_1$ and $L_2$ intersect and find the position vector of the point of intersection.

13. Lines $L_1$ and $L_2$ have vector equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \text{ respectively. Show}$$

that $L_1$ and $L_2$ intersect and find the position vector of the point of intersection.

15. For each of the pairs of lines given by the following vector equations state whether the lines are parallel lines, non-parallel coplanar lines or skew lines.
(a) $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} + \mu(3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$.
(b) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j})$.
(c) $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + 7\mathbf{j} + 6\mathbf{k} + \mu(3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$.
(d) $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = -8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$.

16. Find the acute angle between the lines with vector equations
$\mathbf{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$,
giving your answer to the nearest degree.

17. Find the acute angle between the lines whose equations are
$\frac{x-2}{-4} = \frac{y-3}{3} = \frac{z+1}{-1}$ and $\frac{x-3}{2} = \frac{y-1}{6} = \frac{z+1}{-5}$, giving your answer to the nearest degree.

18. The vector equations of three lines are:
line 1 $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$
line 2 $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \mu(-\mathbf{i} - 2\mathbf{k})$
line 3 $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + \eta(2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$
(a) Show that lines 1 and 2 intersect and find the position vector of the point of intersection.
(b) Show that lines 2 and 3 intersect and find the position vector of the point of intersection.
(c) Find the distance between these two points of intersection.

19. Two lines $L_1$ and $L_2$ lie in the $x$–$y$ plane and have cartesian equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Show that the vector equations of $L_1$ and $L_2$ can be written

$$\mathbf{r}_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ m_1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ m_2 \end{pmatrix}.$$

Use vector methods to show that if $\theta$ is the angle between $L_1$ and $L_2$

then $\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$.

(i.e. obtain the result of page 380 by vector methods).

20. For each of the following parts find the perpendicular distance from the given point to the given line,
(a) the point with position vector $\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ and the line $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$
(b) the point with position vector $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and the line $\mathbf{r} = \mathbf{i} - 6\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$
(c) the point $(1, 1, 3)$ and the line $\frac{x+4}{2} = \frac{y+1}{3} = \frac{z-1}{3}$
(d) the point $(-6, -4, -5)$ and the line $x - 5 = \frac{y-6}{2} = \frac{z-3}{4}$

ANSWERS

*page 420* **Exercise 17C**

**1.** $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ **2.** $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

**3.** $2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ or multiples thereof **5.** B and C **6.** F **7.** 6, 8

**8.** (a) $3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ (b) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$

**9.** (a) $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z+1}{1}$ (b) $\frac{x-3}{3} = \frac{y+1}{2} = \frac{z-2}{-4}$ (c) $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-1}{-1}$

**10.** $\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} - \mathbf{k})$

**11.** (a) $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$ (b) $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

**12.** $-4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ **13.** $\begin{pmatrix} 9 \\ 3 \\ 0 \end{pmatrix}$

**14.** (a) (5, 0, 1) (b) lines do not intersect (c) (4, 5, 9) (d) (12, −3, 3)

**15.** (a) parallel (b) non parallel coplanar (c) skew (d) non parallel coplanar

**16.** 79° **17.** 69° **18.** (a) $\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ (b) $-4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ (c) $5\sqrt{5}$ units

**20.** (a) $\sqrt{5}$ units (b) $3\sqrt{2}$ units (c) $\sqrt{11}$ units (d) $4\sqrt{6}$ units

**21.** (a) $\frac{1}{3}\sqrt{21}$ units (b) $\frac{1}{2}\sqrt{14}$ units

**22.** $\frac{2}{3}\sqrt{35}$ units **23.** (a) $\sqrt{3}$ units (b) $\frac{4}{7}\sqrt{21}$ units

*P4 Book page 213 EX6A Q1alt,2,4,5,6,7alt,9,10

1 Simplify as much as possible:

(a) $3\mathbf{i} \times \mathbf{j}$

(b) $2\mathbf{i} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$

(c) $(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) + (\mathbf{k} + \mathbf{i}) \times (\mathbf{j} - \mathbf{k})$

(d) $(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + \mathbf{k})$

(e) $(2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (-\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

(f) $(\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (3\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

(g) $(2\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \times (-3\mathbf{i} + 4\mathbf{k})$

(h) $(2\mathbf{i} + \mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$

(i) $(2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$

(j) $(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \times (5\mathbf{i} - 4\mathbf{k})$

2 Find a unit vector which is perpendicular to the vector $(4\mathbf{i} + 4\mathbf{j} - 7\mathbf{k})$ and to the vector $(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$. [L]

3 Find a unit vector perpendicular to both $2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

4 Find a vector of magnitude 7 which is perpendicular to both $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

5 Find the magnitude of the vector $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k})$. [L]

6 Given that $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ find

(a) $\mathbf{a}.\mathbf{b}$

(b) $\mathbf{a} \times \mathbf{b}$

(c) the unit vector in the direction of $\mathbf{a} \times \mathbf{b}$. [L]

7 Find the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$ where

(a) $\mathbf{a} = 2\mathbf{i} - \mathbf{j}, \quad \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

(b) $\mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + 3\mathbf{k}$

(c) $\mathbf{a} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

(d) $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$

(e) $\mathbf{a} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

8 Given that $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = p\mathbf{j} + q\mathbf{k}$ and that $\mathbf{a} \times \mathbf{b} = 2\mathbf{j} + \lambda\mathbf{k}$, find the values of the scalar constants $p$, $q$ and $\lambda$.

9 Given that $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = a\mathbf{i} + b\mathbf{k}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{i} + c\mathbf{k}$, find the values of the scalar constants $a$, $b$ and $c$. Find, in surd form, the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

10 Given that $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, $\mathbf{k} \times \mathbf{r} = \mathbf{p}$, $\mathbf{r} \times \mathbf{p} = \mathbf{k}$, where $a$, $b$, $c$ are scalar constants, show that

$$a^2 + b^2 = 1 \quad \text{and} \quad c = 0$$

## Exercise 6A

**1** (a) $3\mathbf{k}$ (b) $-2\mathbf{j}+2\mathbf{k}$

(c) $2\mathbf{k}$ (d) $\mathbf{i}-3\mathbf{j}-5\mathbf{k}$

(e) $10\mathbf{i}+7\mathbf{j}+\mathbf{k}$ (f) $\mathbf{0}$

(g) $4\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ (h) $2\mathbf{i}-5\mathbf{j}-4\mathbf{k}$

(i) $8\mathbf{i}-8\mathbf{j}-8\mathbf{k}$ (j) $-8\mathbf{i}-19\mathbf{j}-10\mathbf{k}$

**2** $\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$ **3** $\frac{1}{7}(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})$

**4** $\frac{7}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ **5** $2\sqrt{2}$

**6** (a) $-14$ (b) $-8(\mathbf{i}+3\mathbf{j}+\mathbf{k})$

(c) $-\frac{1}{\sqrt{11}}(\mathbf{i}+3\mathbf{j}+\mathbf{k})$

**7** (a) $\sqrt{\frac{14}{15}}$ (b) $\sqrt{\frac{23}{55}}$ (c) $\frac{5\sqrt{3}}{9}$

(d) $\frac{3\sqrt{3}}{14}$ (e) $\frac{\sqrt{7}}{14}$

**8** $p=2,\quad q=-2,\quad \lambda=2$

**9** $a=-1,\quad b=-1,\quad c=-1;\quad -\frac{2\sqrt{2}}{3}$

*P4 book page 220 Ex6B Q1,3,5-11,13,14,16

**1** Find the area of the triangle with vertices $A(0, 0, 0)$, $B(1, 2, 1)$ and $C(-1, 3, 3)$.

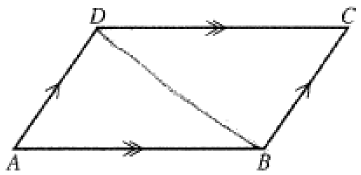
**2** Find the area of the triangle with vertices $A(-5, 1, 4)$, $B(0, 0, 0)$ and $C(-2, 3, -1)$.

**3** Find the area of the triangle with vertices $A(1, -2, 3)$, $B(-1, -1, 4)$ and $C(-2, 1, 5)$.

**4** Find the area of the triangle with vertices $A(2, -1, -1)$, $B(-2, 1, -3)$ and $C(1, -1, 0)$.

**5** Find the area of the triangle with vertices $A(-1, 3, 1)$, $B(2, 2, -3)$ and $C(-1, 3, -4)$.

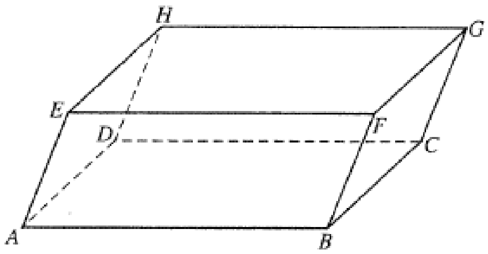
**6** Find the area of the parallelogram $ABCD$ where $A$ is the point with coordinates $(1, 2, -3)$, $B$ is the point with coordinates $(-1, 3, -4)$ and $D$ is the point with coordinates $(1, 5, -2)$.




**7** Find the area of the parallelogram $ABCD$ in which the vertices $A$, $B$ and $D$ have coordinates $(-1, 2, 1)$, $(3, 1, 2)$ and $(5, 1, -6)$ respectively.

**8** Find the area of the triangle with vertices $A(3, -1, 2)$, $B(1, -1, 3)$ and $C(4, -3, 1)$.

**9**




Find the volume of the parallelepiped $ABCDEFGH$ where the vertices $A$, $B$, $D$ and $E$ have coordinates $(0, 0, 0)$, $(5, -2, 3)$, $(2, -3, 4)$ and $(3, -1, -2)$ respectively.

**10** The points $A$, $B$, $C$, $D$ have position vectors

$$\mathbf{a} = (2\mathbf{i} + \mathbf{j}) \qquad \mathbf{b} = (3\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$\mathbf{c} = (-2\mathbf{j} - \mathbf{k}) \qquad \mathbf{d} = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$$

respectively.

(a) Find $\overrightarrow{AB} \times \overrightarrow{BC}$ and $\overrightarrow{BD} \times \overrightarrow{DC}$.

(b) Hence find

(i) the area of $\triangle ABC$

(ii) the volume of the tetrahedron $ABCD$. [L]

**11** Relative to an origin $O$, the points $P$ and $Q$ have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where

$$\mathbf{p} = a(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \quad \text{and} \quad \mathbf{q} = a(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \text{ and } a > 0.$$

Find the area of triangle $OPQ$. [L]

**12** Referred to an origin $O$, the points $P$ and $Q$ have position vectors $3\mathbf{i} - 3\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$ respectively. Find

(a) $\overrightarrow{OP}.\overrightarrow{OQ}$

(b) $\overrightarrow{OP} \times \overrightarrow{OQ}$

(c) the size, in degrees to $0.1°$, of $\angle POQ$

(d) the area of $\triangle OPQ$.

**13** Referred to $O$ as origin, $\overrightarrow{OA} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\overrightarrow{OB} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OC} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

(a) Show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{OC}$.

(b) Find the area of $\triangle OAB$.

(c) Calculate the area of $\triangle ABC$.

**14** The points $A(1, -1, -1)$, $B(-1, 1, -1)$, $C(-1, -1, 1)$ and $D(1, 1, 1)$ are given referred to a fixed origin $O$.

(a) Show that $ABCD$ is a regular tetrahedron.

(b) Find the volume of $ABCD$.

**15** Find the volume of the tetrahedron with vertices at the points $(1, 3, -1)$, $(2, 2, 3)$, $(4, 2, -2)$ and $(3, 7, 4)$.

**16** A tetrahedron has its vertices at the points $O(0, 0, 0)$, $A(-1, 1, 2)$, $B(1, 2, -1)$ and $C(0, 1, 3)$.

(a) Determine the area of the face $ABC$.

(b) Find a unit vector normal to the face $ABC$.

(c) Find the volume of the tetrahedron.

**17** Find the volume of the tetrahedron with vertices $(0, 1, 0)$, $(0, 0, -4)$, $(2, -1, 3)$, $(2, -1, 2)$.

**18** The tetrahedron $ABCD$ has vertices $A(1,-1,0)$, $B(0,2,-1)$, $C(0,2,1)$, $D(-1,3,0)$.

(a) Find the area of face $BCD$.

(b) Find the volume of the tetrahedron.

**19** A tetrahedron $OABC$ has its vertices at the points $O(0,0,0)$, $A(1,2,-1)$, $B(-1,1,2)$ and $C(2,-1,1)$.

(a) Write down expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of **i**, **j** and **k** and find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(b) Deduce the area of $\triangle ABC$.

(c) Find the volume of the tetrahedron $OABC$. [L]

**20** The edges $OP$, $OQ$, $OR$ of a tetrahedron $OPQR$ are the vectors **a**, **b** and **c** respectively, where

$$\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$$
$$\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$
$$\mathbf{c} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

(a) Evaluate $\mathbf{b} \times \mathbf{c}$ and deduce that $OP$ is perpendicular to the plane $OQR$.

(b) Write down the length of $OP$ and the area of $\triangle OQR$ and hence the volume of the tetrahedron.

(c) Verify your result by evaluating $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$. [L]

ANSWERS

## Exercise 6B

**1** $\frac{5\sqrt{2}}{2}$ **2** $\frac{13\sqrt{3}}{2}$ **3** $\frac{\sqrt{11}}{2}$ **4** $\sqrt{11}$

**5** $\frac{5\sqrt{10}}{2}$ **6** $2\sqrt{14}$ **7** $6\sqrt{34}$ **8** $\frac{\sqrt{21}}{2}$

**9** 39

**10** (a) $5\mathbf{i} - \mathbf{j} - 7\mathbf{k}$, $2\mathbf{i} - 8\mathbf{j} + \mathbf{k}$

(b) (i) $\frac{5\sqrt{3}}{2}$ (ii) $\frac{19}{6}$

**11** $\frac{a\sqrt{3}}{2}$

**12** (a) 24 (b) $6\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$

(c) $39.7°$ (d) $3\sqrt{11}$

**13** (b) $\frac{1}{2}\sqrt{171}$ (c) $\frac{3}{2}\sqrt{133}$

**14** (b) $\frac{8}{3}$ **15** 12

**16** (a) $\frac{3}{2}\sqrt{3}$ (b) $\pm\frac{1}{\sqrt{27}}(\mathbf{i} - 5\mathbf{j} - \mathbf{k})$

(c) $\frac{4}{3}$

**17** $\frac{1}{3}$

**18** (a) $\sqrt{2}$ (b) $\frac{2}{3}$

**19** (a) $-2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$; $7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

(b) $\frac{7}{2}\sqrt{3}$ (c) $\frac{7}{3}$

**20** (a) $\mathbf{i} + 2\mathbf{j}$ (b) $2\sqrt{5}$, $\frac{1}{2}\sqrt{5}$; $\frac{5}{3}$

*P4 book page 228 Ex6C Q1ab,2ab,3ab,4ab,5ab,6a,7ac,8ac,9-12

1 Find, in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, an equation of the straight line passing through the point with position vector $\mathbf{a}$ and which is parallel to the vector $\mathbf{b}$ where:

(a) $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(b) $\mathbf{a} = \mathbf{i} - 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$

(c) $\mathbf{a} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

(d) $\mathbf{a} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

2 Find, in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, an equation of the line passing through the points with coordinates

(a) (3, 2, 5), (9, 5, 17)

(b) (1, −2, −1), (2, −4, −4)

(c) (−4, −3, 11), (0, −3, 1)

(d) (3, −2, 3), (−1, 0, 1)

3 Find an equation, in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, of the line given by the equation, where $\lambda$ is a scalar:

(a) $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

(b) $\mathbf{r} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$

(c) $\mathbf{r} = \mathbf{i} - 4\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

(d) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(-3\mathbf{i} + 2\mathbf{j})$

4 Find in the form $\mathbf{r.n} = p$ an equation of the plane that passes through the point with position vector $\mathbf{a}$ and is perpendicular to the vector $\mathbf{n}$ where:

(a) $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{n} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$, $\mathbf{n} = -\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

(c) $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$, $\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

(d) $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{n} = -3\mathbf{i} + \mathbf{k}$

(e) $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{n} = 6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

5 Find a cartesian equation for each of the planes in question 4.

6 Verify that the point with position vector $\mathbf{a}$ lies in the given plane ($\lambda$, $\mu$ scalars) where:

(a) $\mathbf{a} = 2\mathbf{j} + \mathbf{k}$, $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

(b) $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \mu(-\mathbf{i} - \mathbf{j} - \mathbf{k})$

(c) $\mathbf{a} = 24\mathbf{i} + 25\mathbf{j} - 9\mathbf{k}$, $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(7\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}) + \mu(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$

7 Find (i) an equation of the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ (ii) a cartesian equation of the plane passing through the points:

(a) (1, −1, 1), (2, −4, 3), (0, 1, −3)

(b) (4, 7, −1), (1, 1, −4), (2, −2, 3)

(c) (8, 1, −1), (2, 6, −2), (3, −3, 0)

(d) (2, 0, −3), (1, 4, −1), (2, −1, 0)

8 Find a cartesian equation of the plane containing the points:

(a) (1, 1, 1), (2, 1, 0), (2, 2, −1)

(b) (2, 1, −1), (−2, −1, −5), (0, −4, 3)

(c) (1, 1, 2), (3, 4, 1), (−5, 1, −1)

(d) (4, 0, 0), (0, 3, 0), $\left(0, 0, -\frac{1}{2}\right)$

9 Find the coordinates of the point of intersection of the line $l$ and the plane $\Pi$ where

(a) $l$: $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + t(-\mathbf{i} + 2\mathbf{k})$, $t$ scalar

$\Pi$: $\mathbf{r}.(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 4$

(b) $l$: $\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + t(2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$, $t$ scalar

$\Pi$: $\mathbf{r}.(6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = 10$

10 Find an equation of the plane, in the form $\mathbf{r}.\mathbf{n} = p$, which contains the line $l$ and the point with position vector $\mathbf{a}$ where

(a) $l$: $\mathbf{r} = t(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$, $\mathbf{a} = \mathbf{i} + 4\mathbf{k}$

(b) $l$: $\mathbf{r} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k} + t(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$, $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

(c) $l$: $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})$, $\mathbf{a} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

11 Find a cartesian equation of the plane which passes through the origin $O$ and contains the line with equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

12 Referred to an origin $O$, the points $A$, $B$, $C$ have coordinates $(3, 2, 0)$, $(1, 0, 1)$, $(2, 2, 2)$ respectively.

(a) Find a cartesian equation of the plane $ABC$.

(b) Show that $D(4, 4, 1)$ lies in the plane.

(c) Show that $AB$ and $DC$ are parallel.

(d) Find the coordinates of the point where the lines $AC$ and $BD$ meet.

**ANSWERS**

**Exercise 6C**

1 (a) $\mathbf{r} \times (-\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

(b) $\mathbf{r} \times (2\mathbf{i} + 3\mathbf{j}) = 6\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

(c) $\mathbf{r} \times (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 13\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$

(d) $\mathbf{r} \times (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 7\mathbf{i} + 11\mathbf{j} + \mathbf{k}$

2 (a) $\left[\mathbf{r} - \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}\right] \times \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = \mathbf{0}$

(b) $\left[\mathbf{r} - \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}\right] \times \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \mathbf{0}$

(c) $\left[\mathbf{r} - \begin{pmatrix} -4 \\ -3 \\ 11 \end{pmatrix}\right] \times \begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix} = \mathbf{0}$

(d) $\left[\mathbf{r} - \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}\right] \times \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = \mathbf{0}$

3 (a) $\left[\mathbf{r} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right] \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \mathbf{0}$

(b) $\left[\mathbf{r} - \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\right] \times \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix} = \mathbf{0}$

(c) $\left[\mathbf{r} - \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}\right] \times \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \mathbf{0}$

(d) $\left[\mathbf{r} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\right] \times \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} = \mathbf{0}$

4 (a) $\mathbf{r} . (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = -4$

(b) $\mathbf{r} . (-\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = -22$

(c) $\mathbf{r} . (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = 11$

(d) $\mathbf{r} . (-3\mathbf{i} + \mathbf{k}) = -4$

(e) $\mathbf{r} . (6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 14$

5 (a) $2x + 4y - z = -4$

(b) $-x + 3y - 4z = -22$

(c) $2x + 3y - 4z = 11$

(d) $-3x + z = -4$

(e) $3x + 2y - z = 7$

**7** (a) (i) $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$

(ii) $8x + 2y - z = 5$

(b) (i) $\mathbf{r} = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -9 \\ 4 \end{pmatrix}$

(ii) $17x - 6y - 5z = 31$

(c) (i) $\mathbf{r} = \begin{pmatrix} 8 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix}$

(ii) $x + 11y + 49z = -30$

(d) (i) $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$

(ii) $14x + 3y + z = 25$

**8** (a) $x + y + z = 3$

(b) $7x - 6y - 4z = 12$

(c) $3x - 4y - 6z + 13 = 0$

(d) $3x + 4y - 24z = 12$

**9** (a) $(0, -2, 3)$ (b) $(3, 1, 2)$

**10** (a) $\mathbf{r}.(4\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 0$

(b) $\mathbf{r}.\begin{pmatrix} 4 \\ -16 \\ 5 \end{pmatrix} = -10$

(c) $\mathbf{r}.\begin{pmatrix} 9 \\ 3 \\ 5 \end{pmatrix} = -14$

**11** $x - 2y + z = 0$

**12** (a) $4x - 3y + 2x = 6$

(d) $\left(2\frac{1}{2}, 2, 1\right)$

*P4 book page 236 Ex6D Q1ac,2ac,3,4ab,5ab,6,7,8ab,11,13-17,19,20,21

1 Find the distance from the origin to the plane with equation:

(a) $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 5$

(b) $\mathbf{r}.(-3\mathbf{i} + \mathbf{j} + 6\mathbf{k}) = 24$

(c) $\mathbf{r}.(5\mathbf{i} - 10\mathbf{j} + 4\mathbf{k}) = 17$

(d) $\mathbf{r}.(3\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}) = 62$

(e) $x - y + 2z = 15$

**2** Find the distance from the given point to the given plane:

(a) $(1, 2, -3)$, $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 23$

(b) $(1, -3, 2)$, $\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) = 46$

(c) $(4, 1, -7)$, $2x + 6y - 3z = 14$

(d) $(1, -3, 5)$, $\mathbf{r}.(4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}) = 10$

(e) $(4, -8, -1)$, $4x + y - 7z = 42$

**3** Find the position vector of the point where the line with equation $\mathbf{r} = \mathbf{i} + \mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 5\mathbf{k})$ cuts the plane with equation $\mathbf{r}.(2\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2$.

**4** Find, in degrees to 0.1°, the acute angle between the given line and plane:

(a) $\mathbf{r} = \mathbf{i} - 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - 3\mathbf{k})$ and $\mathbf{r}.(2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 10$

(b) $\mathbf{r} = 3\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(4\mathbf{i} - 7\mathbf{k})$ and $\mathbf{r}.(-\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}) = 24$

(c) $\mathbf{r} = -6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(-2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$ and $\mathbf{r}.(2\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}) = 63$

**5** Find, in degrees to 0.1°, the angle between the planes with equations:

(a) $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = 3$ and $\mathbf{r}.(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 6$

(b) $\mathbf{r}.(\mathbf{i} - \mathbf{j} - \sqrt{2}\mathbf{k}) = 5$ and $\mathbf{r}.(7\mathbf{j} + \mathbf{k}) = 10$

(c) $x - 2y - 5z = 7$ and $3x + 7y - z = 4$

**6** Find the distance from the origin to the plane with equation $\mathbf{r}.(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 6$.

**7** Find the cosine of the acute angle between the planes with equations $2x + 3y - 4z = 5$ and $6x - 2y - 3z = 4$.

**8** Find a vector equation of the line of intersection of the planes with equations:

(a) $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-3\mathbf{k}) = 28$ and $\mathbf{r}.(4\mathbf{i}-7\mathbf{j}+\mathbf{k}) = 31$

(b) $x+6y+z=-10$, $3x+2y-z=-1$

(c) $\mathbf{r}.(\mathbf{i}+2\mathbf{j}+\mathbf{k}) = -5$ and $\mathbf{r}.(2\mathbf{i}-3\mathbf{j}+\mathbf{k}) = 15$

**9** Find, in degrees to 0.1°, the acute angle between the planes with equations $3x+4y+2z=7$ and $2x-3y+z=9$.

**10** Find the distance of the origin from the plane with equation $\mathbf{r}.(3\mathbf{i}-4\mathbf{j}-12\mathbf{k}) = 26$.

**11** (a) Find, in cartesian form, the equation of the plane $\Pi$ which passes through the origin and contains the line with equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

(b) Find, in degrees to 0.1°, the acute angle between $\Pi$ and the plane with equation

$$4x+y-z=3$$

**12** A line has equation

$$\mathbf{r} = 3\mathbf{i}-5\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}-4\mathbf{j}+\mathbf{k})$$

and a plane has equation

$$\mathbf{r}.(3\mathbf{i}-\mathbf{j}-5\mathbf{k}) = 1$$

Find the acute angle between the line and the plane.

**13** The planes with equations

$$2x-y+3z+3=0 \quad \text{and} \quad x+10y=21$$

meet in a line $L$.

The planes with equations

$$2x-y=0 \quad \text{and} \quad 7x+z=6$$

meet in a line $M$.

Show that $L$ and $M$ meet at a point. Show further that $L$ and $M$ both lie in the plane with equation $x+3y+z=6$.

**14** Referred to a fixed origin $O$, the lines $l_1$ and $l_2$ have equations

$$\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + \mathbf{k} + s(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

and

$$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

respectively, where $s$ and $t$ are scalar parameters.

(a) Show that $l_1$ and $l_2$ intersect and determine the position vector of their point of intersection.

(b) Show that the vector $-\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ is perpendicular to both $l_1$ and $l_2$.

(c) Find, in the form $\mathbf{r}.\mathbf{n} = p$, an equation of the plane containing $l_1$ and $l_2$. [L]

**15** With respect to the origin $O$ the points $A$, $B$, $C$ have position vectors

$$5\mathbf{i} - \mathbf{j} - 3\mathbf{k}, \quad -4\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \quad 5\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}$$

respectively. Find

(a) a vector equation for the line $BC$

(b) a vector equation for the plane $OAB$

(c) the cosine of the acute angle between the lines $OA$ and $OB$.

Obtain, in the form $\mathbf{r}.\mathbf{n} = p$, a vector equation for $\Pi$, the plane which passes through $A$ and is perpendicular to $BC$.

Find cartesian equations for

(d) the plane $\Pi$

(e) the line $BC$. [L]

**16** Show that

$$\mathbf{r} = \mathbf{a} + s(\mathbf{b} - \mathbf{a}) + t(\mathbf{c} - \mathbf{a})$$

is an equation of the plane which passes through the non-collinear points whose position vectors are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, where $\mathbf{r}$ is the position vector of a general point on the plane and $s$ and $t$ are scalars. Find a cartesian equation of the plane containing the points $(1, 1, -1)$, $(2, 0, 1)$ and $(3, 2, 1)$, and show that the points $(2, 1, 2)$ and $(0, -2, -2)$ are equidistant from, and on opposite sides of, this plane. [L]

**17** A plane passes through the three points $A$, $B$, $C$ whose position vectors, referred to an origin $O$, are $(\mathbf{i}+3\mathbf{j}+3\mathbf{k})$, $(3\mathbf{i}+\mathbf{j}+4\mathbf{k})$, $(2\mathbf{i}+4\mathbf{j}+\mathbf{k})$ respectively. Find, in the form $(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})$, a unit vector normal to this plane. Find also a cartesian equation of the plane, and the perpendicular distance from the origin to this plane. [L]

**18** Show that the vector $\mathbf{i}+\mathbf{k}$ is perpendicular to the plane with vector equation

$$\mathbf{r}=\mathbf{i}+s\mathbf{j}+t(\mathbf{i}-\mathbf{k})$$

Find the perpendicular distance from the origin to this plane. Hence, or otherwise, obtain a cartesian equation of the plane. [L]

**19** Three planes have equations

$$x-6y-z=5$$
$$3x+2y+z=-1$$
$$5x+pz=q$$

Show that

(a) the planes have a common point of intersection unless $p=1$

(b) when $p=1$, $q=2$, the planes intersect in pairs in three parallel lines

(c) when $p=1$, $q=1$, the planes have a common line of intersection.

Give equations for the line of intersection in (c). [L]

**20** Show that the lines $l_1$, $l_2$, with vector equations

$$\mathbf{r}=5\mathbf{i}-2\mathbf{j}+3\mathbf{k}+\lambda(-3\mathbf{i}+\mathbf{j}-\mathbf{k})$$
$$\mathbf{r}=10\mathbf{i}-3\mathbf{j}+6\mathbf{k}+\mu(4\mathbf{i}-\mathbf{j}+2\mathbf{k})$$

respectively, intersect and find a vector equation of the plane $\Pi$ containing $l_1$ and $l_2$.

Show that the point $Q$ with position vector $(6\mathbf{i}+7\mathbf{j}-2\mathbf{k})$ lies on the line which is perpendicular to $\Pi$ and which passes through the intersection of $l_1$ and $l_2$. Find a vector equation of the plane which passes through $Q$ and is parallel to $\Pi$. [L]

**21** With respect to a fixed origin $O$, the straight lines $l_1$ and $l_2$ are given by

$$l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$
$$l_2 : \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \mu(-3\mathbf{i} + 4\mathbf{k})$$

where $\lambda$ and $\mu$ are scalar parameters.

(a) Show that the lines intersect.

(b) Find the position vector of their point of intersection.

(c) Find the cosine of the acute angle contained between the lines.

(d) Find a vector equation of the plane containing the lines. [L]

**22** The position vectors of the points $A$, $B$, $C$ are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where

$$\mathbf{a} = -2\mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}, \quad \mathbf{c} = 5\mathbf{j} - 4\mathbf{k}$$

(a) Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ and hence, or otherwise, find an equation of the plane $ABC$ in the form $\mathbf{r}.\mathbf{n} = p$ and the area of the triangle $ABC$.

(b) Find a vector equation of the plane which passes through $A$ and which is perpendicular to both the plane $ABC$ and the plane with equation $(\mathbf{r} - \mathbf{a}).\mathbf{b} = 0$.

(c) Find the cartesian equations of the line $BC$. [L]

**23** Planes $\Pi_1$ and $\Pi_2$ have equations given by

$$\Pi_1 : \mathbf{r}.(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$$
$$\Pi_2 : \mathbf{r}.(\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 1$$

(a) Show that the point $A(2, -2, 3)$ lies in $\Pi_2$.

(b) Show that $\Pi_1$ is perpendicular to $\Pi_2$.

(c) Find, in vector form, an equation of the straight line through $A$ which is perpendicular to $\Pi_1$.

(d) Determine the coordinates of the point where this line meets $\Pi_1$.

(e) Find the perpendicular distance of $A$ from $\Pi_1$.

(f) Find a vector equation of the plane through $A$ parallel to $\Pi_1$. [L]

**24** Show that the line with equations

$$\frac{x-4}{1} = \frac{y-5}{2} = \frac{z-6}{3}$$

and the line with equations

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}$$

intersect.

Find an equation for the plane in which they lie and the coordinates of their point of intersection. [L]

## Exercise 6D

**1** (a) $\frac{5}{14}\sqrt{14}$ (b) $\frac{12}{23}\sqrt{46}$ (c) $\frac{17}{141}\sqrt{141}$
(d) $\frac{62}{13}$ (e) $\frac{5}{2}\sqrt{6}$

**2** (a) $\frac{5}{2}\sqrt{6}$ (b) $\frac{5}{2}\sqrt{14}$ (c) 3
(d) $3\frac{1}{3}$ (e) $\frac{9}{22}\sqrt{66}$

**3** $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$

**4** (a) 46.4° (b) 40.3° (c) 67.1°

**5** (a) 27.3° (b) 53.5° (c) 81.8°

**6** 2 **7** $\dfrac{18\sqrt{29}}{203}$

**8** (a) $\mathbf{r}=\frac{121}{14}\mathbf{i}-\frac{25}{7}\mathbf{k}+\lambda\left(\frac{10}{7}\mathbf{i}+\mathbf{j}+\frac{9}{7}\mathbf{k}\right)$
(b) $\mathbf{r}=-\frac{11}{8}\mathbf{j}-\frac{7}{4}\mathbf{k}+\lambda\left(\mathbf{i}-\frac{1}{2}\mathbf{j}+2\mathbf{k}\right)$
(c) $\mathbf{r}=20\mathbf{i}-25\mathbf{k}+\lambda(5\mathbf{i}+\mathbf{j}-7\mathbf{k})$

**9** 78.5° **10** 2

**11** (a) $x-2y+z=0$ (b) 84.5°

**12** 10.6°

**14** (a) $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$

(c) $\mathbf{r}\cdot\begin{pmatrix}-1\\3\\7\end{pmatrix}=22$

**15** (a) $\mathbf{r}=\begin{pmatrix}-4\\4\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\-2\\4\end{pmatrix}$

(b) $\mathbf{r}=s\begin{pmatrix}5\\-1\\-3\end{pmatrix}+t\begin{pmatrix}-4\\4\\-1\end{pmatrix}$

(c) $\sqrt{\frac{21}{55}}$; $\mathbf{r}\cdot\begin{pmatrix}3\\-2\\4\end{pmatrix}=5$

(d) $3x-2y+4z=5$

(e) $\dfrac{x+4}{3}=\dfrac{y-4}{-2}=\dfrac{z+1}{4}$

**16** $4x-2y-3z=5$

**17** $\frac{1}{5\sqrt{2}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$; $3x+5y+4z=30$; $3\sqrt{2}$

**18** $\frac{1}{2}\sqrt{2}$; $x+z=1$

**19** (c) $\dfrac{x}{1}=\dfrac{y+1}{1}=\dfrac{z-1}{-5}$

**20** $\mathbf{r}=\begin{pmatrix}2\\-1\\2\end{pmatrix}+s\begin{pmatrix}-3\\1\\-1\end{pmatrix}+t\begin{pmatrix}4\\-1\\2\end{pmatrix}$

$\mathbf{r}=\begin{pmatrix}6\\7\\-2\end{pmatrix}+p\begin{pmatrix}-3\\1\\-1\end{pmatrix}+q\begin{pmatrix}4\\-1\\2\end{pmatrix}$

**21** (b) $7\mathbf{i}+2\mathbf{j}-6\mathbf{k}$ (c) $\frac{14}{15}$

(d) $\mathbf{r}=\begin{pmatrix}7\\2\\-6\end{pmatrix}+s\begin{pmatrix}2\\1\\-2\end{pmatrix}+t\begin{pmatrix}-3\\0\\4\end{pmatrix}$

**22** (a) $4\mathbf{i}+8\mathbf{j}+10\mathbf{k}$; $\mathbf{r}\cdot\begin{pmatrix}4\\8\\10\end{pmatrix}=0$; $3\sqrt{5}$

(b) $\mathbf{r}=\begin{pmatrix}-2\\1\\0\end{pmatrix}+s\begin{pmatrix}4\\8\\10\end{pmatrix}+t\begin{pmatrix}1\\2\\-2\end{pmatrix}$

(c) $\dfrac{x-1}{-1}=\dfrac{y-2}{3}=\dfrac{z+2}{-2}$ $(=\lambda)$

**23** (c) $\mathbf{r}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$

(d) $\left(-1,\ -\frac{1}{2},\ 1\frac{1}{2}\right)$

(e) $\frac{3}{2}\sqrt{6}$

(f) $\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=9$

**24** $\mathbf{r}=\begin{pmatrix}5\\7\\9\end{pmatrix}+s\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}4\\5\\6\end{pmatrix}$; (5, 7, 9)

**Complex Numbers Question Booklet for AS1**

**-P3 Book p166 Ex7A Q1ace,2ace,3ace,4,5ace,7ace,8ace,9ace,11,12 (*AS)**

# Exercise 7A

**1** Express in terms of i:

(a) $\sqrt{(-64)}$ (b) $\sqrt{(-7)}$ (c) $\sqrt{16}-\sqrt{(-81)}$

(d) $3-\sqrt{(-25)}$ (e) $\sqrt{(-100)}-\sqrt{(-49)}$

**2** Simplify:

(a) $i^3$ (b) $i^7$ (c) $i^{-9}$

(d) $i(2i-3i^3)$ (e) $(i+2i^2)(3-i)$

**3** Write in the form $a+ib$, where $a, b \in \mathbb{R}$:

(a) $2i(5-2i)$ (b) $(2+i)^2$ (c) $(4-i)^5$

(d) $(1+2i)^2+(3-i)^3$ (e) $(1+i)^2-3(2-i)^3$

**4** Find $z^*$ given that $z =$

(a) $2+4i$ (b) $3-6i$ (c) $-5+2i$

(d) $-7-3i$ (e) $2i-4$ (f) $6$

(g) $3i$ (h) $-3i+7$

**5** Simplify:

(a) $(2+3i)+(4-7i)$ (b) $(-3+5i)+(-6-7i)$

(c) $(-7-10i)+(2-3i)$ (d) $(2+4i)-(3-6i)$

(e) $(-3+5i)-(-7+4i)$ (f) $(-9-6i)-(-8-9i)$

(g) $(6-3i)-(8-5i)$

**6** Express in the form $a+ib$, where $a, b \in \mathbb{R}$:

(a) $(2+i)(3-i)$ (b) $(-3-4i)(2-7i)$

(c) $(5+2i)(-3+4i)$ (d) $(1-5i)^2$

(e) $(2-i)^3$ (f) $(1+i)(2-i)(i+3)$

(g) $i(3-7i)(2-i)$

**7** Express in the form $a+ib$, where $a, b \in \mathbb{R}$:

(a) $\dfrac{2-7i}{1+2i}$ (b) $\dfrac{1+2i}{3-i}$ (c) $\dfrac{1+2i}{3+4i}$

(d) $\dfrac{1}{1+2i}$ (e) $\dfrac{2+3i}{2-3i}$ (f) $\dfrac{5+i}{i-3}$

(g) $\dfrac{6}{4i-3}$ (h) $\dfrac{1}{(i+2)(1-2i)}$

**8** Solve:

(a) $x^2 + 25 = 0$  (b) $x^3 + 64x = 0$

(c) $x^2 - 4x + 5 = 0$  (d) $x^2 + 6x + 10 = 0$

(e) $x^2 + 29 = 4x$  (f) $2x^2 + 3x + 7 = 0$

(g) $3x^2 + 2x + 1 = 0$  (h) $3x^2 - 2x + 2 = 0$

**9** Express in the form $a + ib$, where $a, b \in \mathbb{R}$:

(a) $\dfrac{1}{1 + 2i} + \dfrac{1}{1 - 2i}$  (b) $\dfrac{1}{2 + i} - \dfrac{1}{1 + 5i}$

(c) $5 - 4i + \dfrac{5}{3 - 4i}$

**10** Given that $z = -1 + 3i$, express $z + \dfrac{2}{z}$ in the form $a + ib$, where $a, b \in \mathbb{R}$.

**11** Given that $T = \dfrac{x - iy}{x + iy}$, where $x$, $y$, $T \in \mathbb{R}$, show that

$$\frac{1 + T^2}{2T} = \frac{x^2 - y^2}{x^2 + y^2}$$

**12** Show that the complex number $\dfrac{2 + 3i}{5 + i}$ can be expressed in the form $\lambda(1 + i)$, where $\lambda$ is real.
State the value of $\lambda$.

Hence, or otherwise, show that $\left(\dfrac{2 + 3i}{5 + i}\right)^4$ is real and determine its value. [L]

## Exercise 7A

**1** (a) $8i$ (b) $i\sqrt{7}$ (c) $4-9i$
(d) $3-5i$ (e) $3i$

**2** (a) $-i$ (b) $-i$ (c) $-i$
(d) $-5$ (e) $-5+5i$

**3** (a) $4+10i$ (b) $3+4i$
(c) $404-1121i$ (d) $15-22i$
(e) $-6+35i$

**4** (a) $2-4i$ (b) $3+6i$ (c) $-5-2i$

(d) $-7+3i$ (e) $-4-2i$ (f) $6$
(g) $-3i$ (h) $7+3i$

**5** (a) $6-4i$ (b) $-9-2i$
(c) $-5-13i$ (d) $-1+10i$
(e) $4+i$ (f) $-1+3i$
(g) $-2+2i$

**6** (a) $7+i$ (b) $-34+34i$
(c) $-23+14i$ (d) $-24-10i$
(e) $2-11i$ (f) $8+6i$
(g) $17-i$

**7** (a) $-\frac{1}{5}(12+11i)$ (b) $\frac{1}{10}(1+7i)$
(c) $\frac{1}{25}(11+2i)$ (d) $\frac{1}{5}(1-2i)$
(e) $\frac{1}{13}(-5+12i)$ (f) $-\frac{1}{5}(7+4i)$
(g) $-\frac{6}{25}(3+4i)$ (h) $\frac{1}{25}(4+3i)$

**8** (a) $\pm 5i$ (b) $0, \pm 8i$ (c) $2 \pm i$
(d) $-3 \pm i$ (e) $2 \pm 5i$
(f) $\frac{1}{4}(-3 \pm i\sqrt{47})$
(g) $\frac{1}{3}(-1 \pm i\sqrt{2})$ (h) $\frac{1}{3}(1 \pm i\sqrt{5})$

**9** (a) $\frac{2}{5}$ (b) $\frac{1}{130}(47-i)$ (c) $\frac{4}{5}(7-4i)$

**10** $-\frac{6}{5}(1-2i)$

**12** $\lambda = \frac{1}{2}; -\frac{1}{4}$

**-P3 Book p179 Ex7C Q1ace,2ace,3,4,8 (*AS)**

# Exercise 7C

1 Find the square roots of:

(a) $5+12i$ (b) $7-24i$ (c) $3-4i$

(d) $-20i$ (e) $1-i4\sqrt{3}$

2 Find the real numbers $x$ and $y$ given that:

(a) $x+4y+xyi = 12-16i$

(b) $2x+(x-2y)i = 18-y-i$

(c) $3x+2xi = 7+2y+(12+5y)i$

(d) $x-7y+8xi = 6y+(6y-100)i$

(e) $2x-y+(y-4)i = 0$

3 Given that $(1+5i)A-2B = 3+7i$, find $A$ and $B$ if:

(a) $A$ and $B$ are real,

(b) $A$ and $B$ are conjugate complex numbers.

4 Given that $x, y \in \mathbb{R}$ and

$$(x+iy)(2+i) = 3-i$$

find $x$ and $y$.

8 Given that

$$\frac{1}{x+iy}+\frac{1}{1+2i} = 1$$

where $x$ and $y$ are real, find $x$ and $y$. [L]

ANSWERS

## Exercise 7C

1 (a) $\pm(3+2i)$ (b) $\pm(4-3i)$ (c) $\pm(2-i)$

(d) $\pm\sqrt{10}(1-i)$ (e) $\pm(2-i\sqrt{3})$

2 (a) $(-4,4)$, $(16,-1)$

(b) $(7,4)$ (c) $(1,-2)$

(d) $(-\frac{650}{49}, -\frac{50}{49})$ (e) $(2,4)$

3 (a) $A=\frac{7}{5}$, $B=-\frac{4}{5}$

(b) $A=2-i$, $B=2+i$

4 $x=1$, $y=-1$

5 (a) $p=3$, $q=1$ (b) $p=-\frac{2}{5}$, $q=\frac{6}{5}$

6 $x=-\frac{7}{25}$, $y=-\frac{24}{25}$ 7 $x=\frac{2}{13}$, $y=\frac{3}{13}$

8 $x=1$, $y=-\frac{1}{2}$ 9 $a=0$, $b=\pm 2$

-**Edexcel Book Ex1F Q1ace,2ace,3,4,8    (*AS)**

## Exercise 1F

1 $f(z) = z^3 - 6z^2 + 21z - 26$

a Show that $f(2) = 0$. **(1 mark)**

b Hence solve $f(z) = 0$ completely. **(3 marks)**

2 $f(z) = 2z^3 + 5z^2 + 9z - 6$

a Show that $f(\frac{1}{2}) = 0$. **(1 mark)**

b Hence write $f(z)$ in the form $(2z - 1)(z^2 + bz + c)$, where $b$ and $c$ are real constants to be found. **(2 marks)**

c Use algebra to solve $f(z) = 0$ completely. **(2 marks)**

3 $g(z) = 2z^3 - 4z^2 - 5z - 3$

Given that $z = 3$ is a root of the equation $g(z) = 0$, solve $g(z) = 0$ completely. **(4 marks)**

4 $p(z) = z^3 + 4z^2 - 15z - 68$

Given that $z = -4 + i$ is a solution to the equation $p(z) = 0$,

a show that $z^2 + 8z + 17$ is a factor of $p(z)$. **(2 marks)**

b Hence solve $p(z) = 0$ completely. **(2 marks)**

5 $f(z) = z^3 + 9z^2 + 33z + 25$

Given that $f(z) = (z + 1)(z^2 + az + b)$, where $a$ and $b$ are real constants,

a find the value of $a$ and the value of $b$ **(2 marks)**

b find the three roots of $f(z) = 0$ **(4 marks)**

c find the sum of the three roots of $f(z) = 0$. **(1 mark)**

6 $g(z) = z^3 - 12z^2 + cz + d = 0$, where $c, d \in \mathbb{R}$.

Given that 6 and $3 + i$ are roots of the equation $g(z) = 0$,

a write down the other complex root of the equation **(1 mark)**

b find the value of $c$ and the value of $d$. **(4 marks)**

7 $h(z) = 2z^3 + 3z^2 + 3z + 1$

Given that $2z + 1$ is a factor of $h(z)$, find the three roots of $h(z) = 0$. **(4 marks)**

8 $f(z) = z^3 - 6z^2 + 28z + k$

Given that $f(2) = 0$,

a find the value of $k$ **(1 mark)**

b find the other two roots of the equation. **(4 marks)**

9 Find the four roots of the equation $z^4 - 16 = 0$.

10 $f(z) = z^4 - 12z^3 + 31z^2 + 108z - 360$

a Write $f(z)$ in the form $(z^2 - 9)(z^2 + bz + c)$, where $b$ and $c$ are real constants to be found. **(2 marks)**

b Hence find all the solutions to $f(z) = 0$. **(3 marks)**

**11** $g(z) = z^4 + 2z^3 - z^2 + 38z + 130$

Given that $g(2 + 3i) = 0$, find all the roots of $g(z) = 0$.

**12** $f(z) = z^4 - 10z^3 + 71z^2 + Qz + 442$, where $Q$ is a real constant.

Given that $z = 2 - 3i$ is a root of the equation $f(z) = 0$,

**a** show that $z^2 - 6z + 34$ is a factor of $f(z)$ **(4 marks)**

**b** find the value of $Q$ **(1 mark)**

**c** solve completely the equation $f(z) = 0$. **(2 marks)**

ANSWERS

## Exercise 1F

**1** **a** $f(2) = 8 - 24 + 42 - 26 = 0$

**b** $z = 2, z = 2 + 3i$ or $z = 2 - 3i$

**2** **a** Substitute $z = \frac{1}{2}$ into $f(z)$.

**b** $b = 3, c = 6$

**c** $z = \frac{1}{2}$, or $z = -\frac{3}{2} \pm \frac{\sqrt{15}}{2}i$

**3** $3, -\frac{1}{2} + \frac{1}{2}i$ and $-\frac{1}{2} - \frac{1}{2}i$

**4** **a** $(z - (-4 + i))(z - (-4 - i)) = z^2 + 8z + 16 + 1$

$= z^2 + 8z + 17$

**b** $z = 4, z = -4 + i$ or $z = -4 - i$

**5** **a** $a = 8, b = 25$ **b** $-1, -4 + 3i, -4 - 3i$ **c** $-9$

**6** **a** $3 - i$ **b** $c = 46, d = -60$

**7** $-\frac{1}{2}, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

**8** **a** $k = -40$ **b** $2 - 4i, 2 + 4i$

**9** $2, -2, 2i$ and $-2i$

**10** **a** $(z^2 - 9)(z^2 - 12z + 40)$ **b** $z = \pm 3, 6 \pm 2i$

**11** $-3 + i, -3 - i, 2 + 3i$ and $2 - 3i$

$(z - (2 - 3i))(z - (2 + 3i)) = z^2 - 4z + 13$

**12** **a** $(z^2 - 4z + 13)(z^2 + bz + c)$

$= z^4 - 10z^3 + 71z^2 + Qz + 442$

$b = -6, c = 34$

**b** $Q = -214$ **c** $z = 2 + 3i, 2 - 3i, 3 + 5i$ or $3 - 5i$

**-Edexcel Book Mixed Exercise 1**

## Mixed exercise 1

**1** Given that $z_1 = 8 - 3\text{i}$ and $z_2 = -2 + 4\text{i}$, find, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

**a** $z_1 + z_2$

**b** $3z_2$

**c** $6z_1 - z_2$

**2** The equation $z^2 + bz + 14 = 0$, where $b \in \mathbb{R}$ has no real roots.
Find the range of possible values of $b$. **(3 marks)**

**3** The solutions to the quadratic equation $z^2 - 6z + 12 = 0$ are $z_1$ and $z_2$.
Find $z_1$ and $z_2$, giving each answer in the form $a \pm \text{i}\sqrt{b}$.

**4** By using the binomial expansion, or otherwise, show that $(1 + 2\text{i})^5 = 41 - 38\text{i}$. **(3 marks)**

**5** $\text{f}(z) = z^2 - 6z + 10$
Show that $z = 3 + \text{i}$ is a solution to $\text{f}(z) = 0$. **(2 marks)**

**6** $z_1 = 4 + 2\text{i}$, $z_2 = -3 + \text{i}$
Express, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

**a** $z_1^*$ **b** $z_1 z_2$ **c** $\dfrac{z_1}{z_2}$

**7** Write $\dfrac{(7 - 2\text{i})^2}{1 + \text{i}\sqrt{3}}$ in the form $x + \text{i}y$ where $x, y \in \mathbb{R}$.

**8** Given that $\dfrac{4 - 7\text{i}}{z} = 3 + \text{i}$, find $z$ in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$. **(2 marks)**

**9** $z = \dfrac{1}{2 + \text{i}}$
Express in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

**a** $z^2$ **b** $z - \dfrac{1}{z}$

10 Given that $z = a + b\text{i}$, show that $\frac{z}{z^*} = \left(\frac{a^2 - b^2}{a^2 + b^2}\right) + \left(\frac{2ab}{a^2 + b^2}\right)\text{i}$ **(4 marks)**

11 The complex number $z$ is defined by $z = \frac{3 + q\text{i}}{q - 5\text{i}}$, where $q \in \mathbb{R}$.

Given that the real part of $z$ is $\frac{1}{13}$,

a find the possible values of $q$ **(4 marks)**

b write the possible values of $z$ in the form $a + b\text{i}$, where $a$ and $b$ are real constants. **(1 mark)**

12 Given that $z = x + \text{i}y$, find the value of $x$ and the value of $y$ such that $z + 4\text{i}z^* = -3 + 18\text{i}$ where $z^*$ is the complex conjugate of $z$. **(5 marks)**

13 $z = 9 + 6\text{i}$, $w = 2 - 3\text{i}$

Express $\frac{z}{w}$ in the form $a + b\text{i}$, where $a$ and $b$ are real constants.

14 The complex number $z$ is given by $z = \frac{q + 3\text{i}}{4 + q\text{i}}$ where $q$ is an integer.

Express $z$ in the form $a + b\text{i}$ where $a$ and $b$ are rational and are given in terms of $q$. **(4 marks)**

15 Given that $6 - 2\text{i}$ is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation **(1 mark)**

b find the quadratic equation, giving your answer in the form $z^2 + bz + c = 0$ where $b$ and $c$ are real constants. **(2 marks)**

16 Given that $z = 4 - k\text{i}$ is a root of the equation $z^2 - 2mz + 52 = 0$, where $k$ and $m$ are positive real constants, find the value of $k$ and the value of $m$. **(4 marks)**

17 $\text{h}(z) = z^3 - 11z + 20$

Given that $2 + \text{i}$ is a root of the equation $\text{h}(z) = 0$, solve $\text{h}(z) = 0$ completely. **(4 marks)**

18 $\text{f}(z) = z^3 + 6z + 20$

Given that $\text{f}(1 + 3\text{i}) = 0$, solve $\text{f}(z) = 0$ completely. **(4 marks)**

19 $\text{f}(z) = z^3 + 3z^2 + kz + 48$, $k \in \mathbb{R}$

Given that $\text{f}(4\text{i}) = 0$,

a find the value of $k$ **(2 marks)**

b find the other two roots of the equation. **(3 marks)**

20 $\text{f}(z) = z^4 - z^3 - 16z^2 - 74z - 60$

a Write $\text{f}(z)$ in the form $(z^2 - 5z - 6)(z^2 + bz + c)$, where $b$ and $c$ are real constants to be found. **(2 marks)**

b Hence find all the solutions to $\text{f}(z) = 0$. **(3 marks)**

21 $\text{g}(z) = z^4 - 6z^3 + 19z^2 - 36z + 78$

Given that $\text{g}(3 - 2\text{i}) = 0$, find all the roots of $\text{g}(z) = 0$. **(4 marks)**

22 $\text{f}(z) = z^4 - 2z^3 - 5z^2 + pz + 24$

Given that $\text{f}(4) = 0$,

a find the value of $p$ **(1 mark)**

b solve completely the equation $\text{f}(z) = 0$. **(5 marks)**

## Mixed exercise

1 a $6 + \mathrm{i}$ b $-6 + 12\mathrm{i}$ c $50 - 22\mathrm{i}$

2 $-2\sqrt{14} < b < 2\sqrt{14}$

3 $3 + \mathrm{i}\sqrt{3}, 3 - \mathrm{i}\sqrt{3}$

4 $(1 + 2\mathrm{i})^5$
$= 1^5 + 5(1)^4(2\mathrm{i}) + 10(1)^3(2\mathrm{i})^2 + 10(1)^2(2\mathrm{i})^3 + 5(1)(2\mathrm{i})^4 + (2\mathrm{i})^5$
$= 1 + 10\mathrm{i} + 40\mathrm{i}^2 + 80\mathrm{i}^3 + 80\mathrm{i}^4 + 32\mathrm{i}^5$
$= 1 + 10\mathrm{i} - 40 - 80\mathrm{i} + 80 + 32\mathrm{i}$
$= 41 - 38\mathrm{i}$

5 Substitute $z = 3 + \mathrm{i}$ into $\mathrm{f}(z)$ to get $\mathrm{f}(z) = 0$.

6 a $4 - 2\mathrm{i}$ b $-14 - 2\mathrm{i}$ c $-1 - \mathrm{i}$

7 $\dfrac{(45 - 28\mathrm{i})(1 - \mathrm{i}\sqrt{3})}{(1 + \sqrt{3}\,\mathrm{i})(1 - \sqrt{3}\,\mathrm{i})} = \dfrac{45 - 28\sqrt{3}}{4} + \left(\dfrac{-45\sqrt{3} - 28}{4}\right)\mathrm{i}$

8 $\dfrac{4 - 7\mathrm{i}}{3 + \mathrm{i}} = \dfrac{(4 - 7\mathrm{i})(3 - \mathrm{i})}{(3 + \mathrm{i})(3 - \mathrm{i})} = \dfrac{12 - 25\mathrm{i} + 7\mathrm{i}^2}{10} = \frac{1}{2} - \frac{5}{2}\mathrm{i}$

9 a $\frac{3}{25} - \frac{4}{25}\mathrm{i}$ b $\frac{-8}{5} - \frac{6}{5}\mathrm{i}$

10 $\dfrac{z}{z^*} = \dfrac{(a + b\mathrm{i})(a + b\mathrm{i})}{(a - b\mathrm{i})(a + b\mathrm{i})} = \dfrac{a^2 + 2ab\mathrm{i} + b^2\mathrm{i}^2}{a^2 - b^2\mathrm{i}^2}$
$= \dfrac{a^2 - b^2}{a^2 + b^2} + \left(\dfrac{2ab}{a^2 + b^2}\right)\mathrm{i}$

11 a $\dfrac{3 + q\mathrm{i}}{q - 5\mathrm{i}} \times \dfrac{q + 5\mathrm{i}}{q + 5\mathrm{i}} = \dfrac{3q - 5q}{q^2 + 25} + \dfrac{q^2 + 15}{q^2 + 25}\mathrm{i}$
$\dfrac{-2q}{q^2 + 25} = \frac{1}{13} \Rightarrow q^2 + 26q + 25 = 0 \Rightarrow q = -1, q = -25$

b $\frac{1}{13} + \frac{8}{13}\mathrm{i}, \frac{1}{13} + \frac{64}{65}\mathrm{i}$

12 $x + y\mathrm{i} + 4\mathrm{i}(x - y\mathrm{i}) = -3 + 18\mathrm{i}$
$(x + 4y) + (4x + y)\mathrm{i} = -3 + 18\mathrm{i}$
$x + 4y = -3, 4x + y = 18 \Rightarrow x = 5, y = -2$

13 $\dfrac{(9 + 6\mathrm{i})(2 + 3\mathrm{i})}{(2 - 3\mathrm{i})(2 + 3\mathrm{i})} = \dfrac{18 + 39\mathrm{i} + 18\mathrm{i}^2}{4 - 9\mathrm{i}^2} = 3\mathrm{i}$

14 $\dfrac{(q + 3\mathrm{i})(4 - q\mathrm{i})}{(4 + q\mathrm{i})(4 - q\mathrm{i})} = \dfrac{7q}{q^2 + 16} + \dfrac{12 - q^2}{q^2 + 16}\mathrm{i}$

15 a $6 + 2\mathrm{i}$ b $z^2 - 12z + 40$

16 $k = 6, m = 4$

17 $z = 2 + \mathrm{i}, 2 - \mathrm{i}$ or $-4$

18 $z = -2, 1 + 3\mathrm{i}$ or $1 - 3\mathrm{i}$

19 a $k = 16$ b $-4\mathrm{i}$ and $-3$

20 a $b = 4, c = 10$ b $z = 6, -1, -2 + \sqrt{6}\,\mathrm{i}$ or $-2 - \sqrt{6}\,\mathrm{i}$

21 $3 - 2\mathrm{i}, 3 + 2\mathrm{i}, \mathrm{i}\sqrt{6}$ and $-\mathrm{i}\sqrt{6}$

22 a $p = -18$ b $1, 4, -\frac{3}{2} + \dfrac{\sqrt{15}}{2}\mathrm{i}$ and $-\frac{3}{2} - \dfrac{\sqrt{15}}{2}\mathrm{i}$

-Edexcel Book Ex2A Q2-6

using GeoGebra.

## Exercise 2A

1 Show these numbers on an Argand diagram.

a $7 + 2i$ b $5 - 4i$ c $-6 - i$ d $-2 + 5i$

e $3i$ f $\sqrt{2} + 2i$ g $-\frac{1}{2} + \frac{5}{2}i$ h $-4$

2 $z_1 = 11 + 2i$ and $z_2 = 2 + 4i$. Show $z_1$, $z_2$ and $z_1 + z_2$ on an Argand diagram.

3 $z_1 = -3 + 6i$ and $z_2 = 8 - i$. Show $z_1$, $z_2$ and $z_1 + z_2$ on an Argand diagram.

4 $z_1 = 8 + 4i$ and $z_2 = 6 + 7i$. Show $z_1$, $z_2$ and $z_1 - z_2$ on an Argand diagram.

5 $z_1 = -6 - 5i$ and $z_2 = -4 + 4i$. Show $z_1$, $z_2$ and $z_1 - z_2$ on an Argand diagram.

6 $z_1 = 7 - 5i$, $z_2 = a + bi$ and $z_3 = -3 + 2i$ where $a, b \in \mathbb{Z}$. Given that $z_3 = z_1 + z_2$,

a find the values of $a$ and $b$ b show $z_1$, $z_2$ and $z_3$ on an Argand diagram.

7 $z_1 = p + qi$, $z_2 = 9 - 5i$ and $z_3 = -8 + 5i$ where $p, q \in \mathbb{Z}$. Given that $z_3 = z_1 + z_2$,

a find the values of $p$ and $q$ b show $z_1$, $z_2$ and $z_3$ on an Argand diagram.

8 The solutions to the quadratic equation $z^2 - 6z + 10 = 0$ are $z_1$ and $z_2$.

a Find $z_1$ and $z_2$, giving your answers in the form $p \pm qi$, where $p$ and $q$ are integers. **(3 marks)**

b Show, on an Argand diagram, the points representing the complex numbers $z_1$ and $z_2$. **(2 marks)**

9 $f(z) = 2z^3 - 19z^2 + 64z - 60$

a Show that $f\left(\frac{3}{2}\right) = 0$. **(1 mark)**

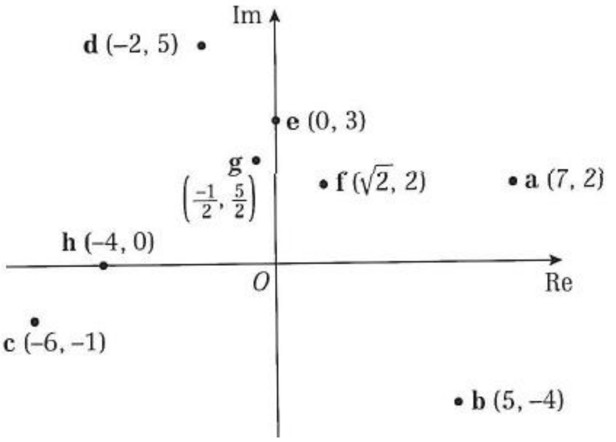
b Use algebra to solve $f(z) = 0$ completely. **(4 marks)**

c Show all three solutions on an Argand diagram. **(2 marks)**

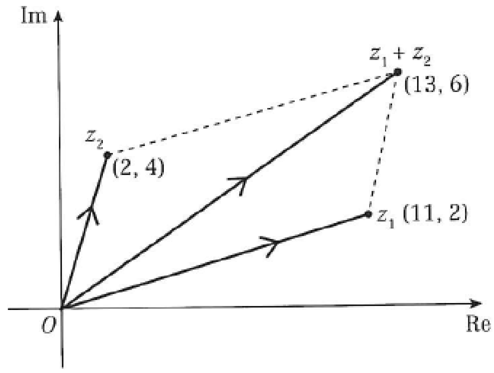
ANSWERS

**Exercise 2A**

1




2

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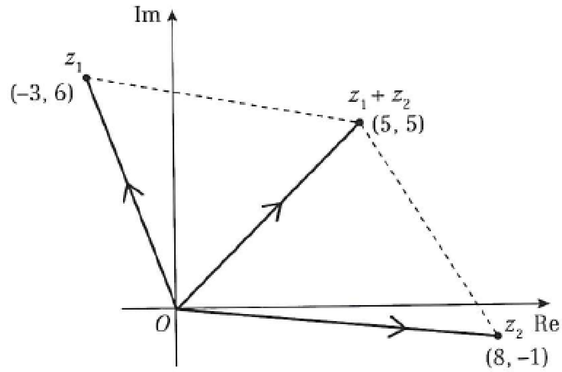

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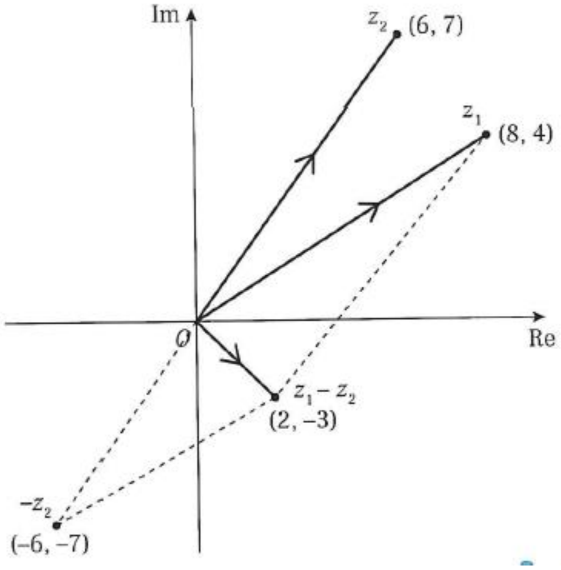


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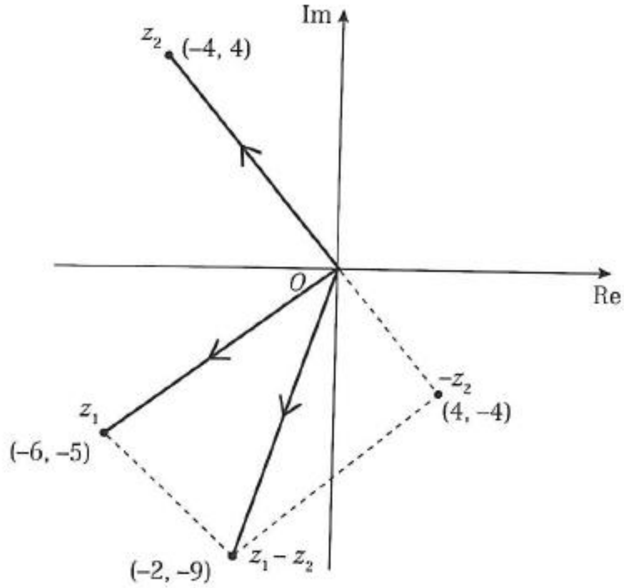
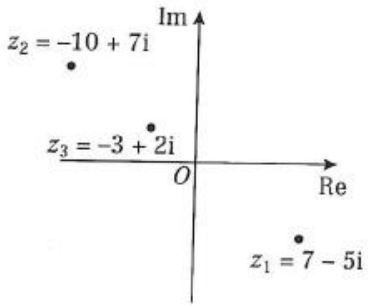


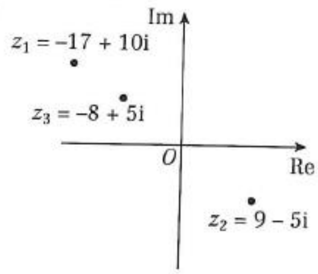

6 a $a = -10$, $b = 7$

b



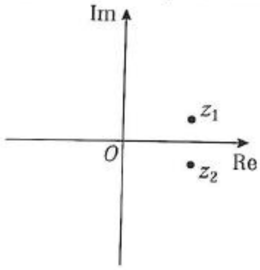

7 a $p = -17$, $q = 10$

b




8 a $z_1 = 3 + i$ and $z_2 = 3 - i$. Other way round acceptable.

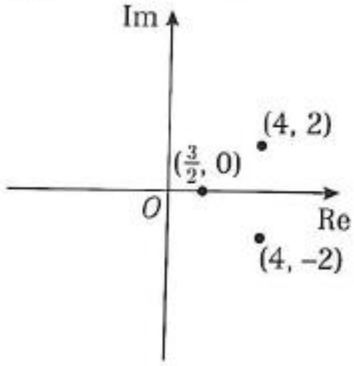
b




9 a $2\left(\frac{3}{2}\right)^3 - 19\left(\frac{3}{2}\right)^2 + 64\left(\frac{3}{2}\right) - 60 = 0$

b $\left(\frac{3}{2}\right)$, $4 + 2i$, $4 - 2i$.

c



-Edexcel Book Ex2C Q1-3(alt parts), Q4-6

## Exercise 2C

1 Express the following in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$. Give the exact values of $r$ and $\theta$ where possible, or values to 2 decimal places otherwise.

a $2 + 2\text{i}$  b $3\text{i}$  c $-3 + 4\text{i}$  d $1 - \sqrt{3}\text{i}$

e $-2 - 5\text{i}$  f $-20$  g $7 - 24\text{i}$  h $-5 + 5\text{i}$

2 Express these in the form $r(\cos\theta + \text{i}\sin\theta)$, giving exact values of $r$ and $\theta$ where possible, or values to two decimal places otherwise.

a $\dfrac{3}{1 + \text{i}\sqrt{3}}$  b $\dfrac{1}{2 - \text{i}}$  c $\dfrac{1 + \text{i}}{1 - \text{i}}$

3 Express the following in the form $x + \text{i}y$, where $x, y \in \mathbb{R}$.

a $5\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$  b $\frac{1}{2}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)$  c $6\left(\cos\frac{5\pi}{6} + \text{i}\sin\frac{5\pi}{6}\right)$

d $3\left(\cos\left(-\frac{2\pi}{3}\right) + \text{i}\sin\left(-\frac{2\pi}{3}\right)\right)$  e $2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$  f $-4\left(\cos\frac{7\pi}{6} + \text{i}\sin\frac{7\pi}{6}\right)$

4 a Express the complex number $z = 4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ in the form $x + iy$, where $x, y \in \mathbb{R}$. **(2 marks)**

b Show the complex number $z$ on an Argand diagram. **(1 mark)**

5 The complex number $z$ is such that $|z| = 7$ and $\arg z = \frac{11\pi}{6}$. Find $z$ in the form $p + q\text{i}$, where $p$ and $q$ are exact real numbers to be found. **(3 marks)**

6 The complex number $z$ is such that $|z| = 5$ and $\arg z = -\frac{4\pi}{3}$. Find $z$ in the form $a + b\text{i}$, where $a$ and $b$ are exact real numbers to be found. **(3 marks)**

ANSWERS

## Exercise 2C

1 a $2\sqrt{2}\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$  b $3\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$

c $5(\cos 2.21 + \text{i}\sin 2.21)$  d $2\left(\cos -\frac{\pi}{3} + \text{i}\sin -\frac{\pi}{3}\right)$

e $\sqrt{29}(\cos(-1.95) + \text{i}\sin(-1.95))$

f $20(\cos\pi + \text{i}\sin\pi)$

g $25(\cos(-1.29) + \text{i}\sin(-1.29))$

h $5\sqrt{2}\left(\cos\frac{3\pi}{4} + \text{i}\sin\frac{3\pi}{4}\right)$

2 a $\frac{3}{2}\left(\cos\left(-\frac{\pi}{3}\right) + \text{i}\sin\left(-\frac{\pi}{3}\right)\right)$  b $\frac{\sqrt{5}}{5}(\cos 0.46 + \text{i}\sin 0.46)$

c $1\left(\cos\frac{\pi}{2} + \text{i}\sin\frac{\pi}{2}\right)$

3 a $5\text{i}$  b $\frac{\sqrt{3}}{4} + \frac{1}{4}\text{i}$  c $-3\sqrt{3} + 3\text{i}$

d $-\frac{3}{2} - \frac{3\sqrt{3}}{2}\text{i}$  e $2 - 2\text{i}$  f $2\sqrt{3} + 2\text{i}$

4 a $-2 + 2\text{i}\sqrt{3}$

b $-2 + 2\text{i}\sqrt{3}$ shown on an Argand diagram.

5 $p = \frac{7\sqrt{3}}{2}$, $q = -\frac{7}{2}$

6 $a = -\frac{5}{2}$, $b = \frac{5\sqrt{3}}{2}$

-Edexcel Book Ex2D Q1a,23ace,4ac

## Exercise 2D

**1** For each given $z_1$ and $z_2$, find the following in the form $r(\cos\theta + \mathrm{i}\sin\theta)$:

**i** $|z_1 z_2|$ **ii** $\arg(z_1 z_2)$ **iii** $z_1 z_2$

**a** $z_1 = 5\left(\cos\frac{3\pi}{8} + \mathrm{i}\sin\frac{3\pi}{8}\right)$, $z_2 = 6\left(\cos\frac{7\pi}{8} + \mathrm{i}\sin\frac{7\pi}{8}\right)$

**b** $z_1 = \sqrt{2}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$, $z_2 = 4\sqrt{2}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$

**2** Given $z_1 = 8\left(\cos\frac{8\pi}{5} + \mathrm{i}\sin\frac{8\pi}{5}\right)$ and $z_2 = 4\left(\cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}\right)$, write down the modulus and argument of:

**a** $z_1 z_2$ **b** $\frac{z_1}{z_2}$ **c** $z_1^2$

**3** Express the following in the form $x + \mathrm{i}y$:

**a** $(\cos 2\theta + \mathrm{i}\sin 2\theta)(\cos 3\theta + \mathrm{i}\sin 3\theta)$

**b** $\left(\cos\frac{3\pi}{11} + \mathrm{i}\sin\frac{3\pi}{11}\right)\left(\cos\frac{8\pi}{11} + \mathrm{i}\sin\frac{8\pi}{11}\right)$

**c** $3\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) \times 2\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

**d** $\sqrt{6}\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right) \times \sqrt{3}\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$

**e** $4\left(\cos\frac{5\pi}{9} - \mathrm{i}\sin\frac{5\pi}{9}\right) \times \frac{1}{2}\left(\cos\frac{5\pi}{18} - \mathrm{i}\sin\frac{5\pi}{18}\right)$

**f** $6\left(\cos\frac{\pi}{10} + \mathrm{i}\sin\frac{\pi}{10}\right) \times 5\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right) \times \frac{1}{3}\left(\cos\frac{2\pi}{5} + \mathrm{i}\sin\frac{2\pi}{5}\right)$

**g** $(\cos 4\theta + \mathrm{i}\sin 4\theta)(\cos\theta - \mathrm{i}\sin\theta)$

**h** $3\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right) \times \sqrt{2}\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)$

**Hint** First make sure both numbers are in modulus–argument form.

**4** Express the following in the form $x + \mathrm{i}y$:

**a** $\dfrac{\cos 5\theta + \mathrm{i}\sin 5\theta}{\cos 2\theta + \mathrm{i}\sin 2\theta}$ **b** $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)}{\frac{1}{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)}$ **c** $\dfrac{3\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)}{4\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)}$ **d** $\dfrac{\cos 2\theta - \mathrm{i}\sin 2\theta}{\cos 3\theta + \mathrm{i}\sin 3\theta}$

**5** $z = -9 + 3\mathrm{i}\sqrt{3}$

**a** Express $z$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \leqslant \pi$ **(2 marks)**

**b** Given that $|w| = \sqrt{3}$ and $\arg w = \frac{7\pi}{12}$, express in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, $-\pi < \theta \leqslant \pi$:

**i** $w$ **ii** $zw$ **iii** $\frac{z}{w}$ **(4 marks)**

ANSWERS

**Exercise 2D**

**1 a i** $|z_1 z_2| = 30$ **ii** $\arg(z_1 z_2) = \frac{5\pi}{4}$

**iii** $30\left(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}\right)$

**b i** $|z_1 z_2| = 8$ **ii** $\arg(z_1 z_2) = \frac{13\pi}{12}$

**iii** $8\left(\cos\frac{13\pi}{12} + \mathrm{i}\sin\frac{13\pi}{12}\right)$

**2 a** $|z_1 z_2| = 32$, $\arg(z_1 z_2) = \frac{4\pi}{15}$

**b** $\left|\frac{z_1}{z_2}\right| = 2$, $\arg\left(\frac{z_1}{z_2}\right) = \frac{14\pi}{15}$

**c** $|z_1^2| = 64$, $\arg(z_1^2) = -\frac{4\pi}{5}$

**3 a** $\cos 5\theta + \mathrm{i}\sin 5\theta$ **b** $-1$ **c** $-\frac{3}{4}\mathrm{i}$

**d** $3\sqrt{2}$ **e** $-\sqrt{3} - \mathrm{i}$ **f** $-5\sqrt{3} + 5\mathrm{i}$

**g** $\cos 3\theta + \mathrm{i}\sin 3\theta$ **h** $3 - 3\mathrm{i}$

**4 a** $\cos 3\theta + \mathrm{i}\sin 3\theta$ **b** $2 + 2\mathrm{i}$ **c** $-\frac{11}{4}\mathrm{i}$

**d** $\cos(-5\theta) + \mathrm{i}\sin(-5\theta)$ or $\cos 5\theta - \mathrm{i}\sin 5\theta$

**5 a** $z = 6\sqrt{3}\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)$

**b i** $w = \sqrt{3}\left(\cos\frac{7\pi}{12} + \mathrm{i}\sin\frac{7\pi}{12}\right)$

**ii** $18\cos\left(-\frac{7\pi}{12}\right) + \mathrm{i}\sin\left(-\frac{7\pi}{12}\right)$

**iii** $6\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$

-Edexcel Book Ex2E Q1acei,4,6acei,7,10ace

## Exercise 2E

1 Sketch the locus of $z$ and give the Cartesian equation of the locus of $z$ when:

a $|z| = 6$ b $|z| = 10$ c $|z - 3| = 2$

d $|z + 3i| = 3$ e $|z - 4i| = 5$ f $|z + 1| = 1$

g $|z - 1 - i| = 5$ h $|z + 3 + 4i| = 4$ i $|z - 5 + 6i| = 5$

**Hint** You may choose a geometric or an algebraic approach to answer these questions.

2 Given that $z$ satisfies $|z - 5 - 4i| = 8$,

a sketch the locus of $z$ on an Argand diagram

b find the exact values of $z$ that satisfy:

i both $|z - 5 - 4i| = 8$ and $\text{Re}(z) = 0$ ii both $|z - 5 - 4i| = 8$ and $\text{Im}(z) = 0$

3 A complex number $z$ is represented by the point $P$ on the Argand diagram. Given that $|z - 5 + 7i| = 5$,

a sketch the locus of $P$

b find the Cartesian equation of this locus

c find the maximum value of $\arg z$ in the interval $(-\pi, \pi)$.

4 On an Argand diagram the point $P$ represents the complex number $z$. Given that $|z - 4 - 3i| = 8$,

a find the Cartesian equation for the locus of $P$ **(2 marks)**

b sketch the locus of $P$ **(2 marks)**

c find the maximum and minimum values of $|z|$ for points on this locus. **(2 marks)**

5 The point $P$ represents a complex number $z$ on an Argand diagram. Given that $|z + 2 - 2\sqrt{3}i| = 2$,

a sketch the locus of $P$ on an Argand diagram **(2 marks)**

b write down the minimum value of $\arg z$ **(2 marks)**

c find the maximum value of $\arg z$. **(2 marks)**

6 Sketch the locus of $z$ and give the Cartesian equation of the locus of $z$ when:

a $|z - 6| = |z - 2|$ b $|z + 8| = |z - 4|$

c $|z| = |z + 6i|$ d $|z + 3i| = |z - 8i|$

e $|z - 2 - 2i| = |z + 2 + 2i|$ f $|z + 4 + i| = |z + 4 + 6i|$

g $|z + 3 - 5i| = |z - 7 - 5i|$ h $|z + 4 - 2i| = |z - 8 + 2i|$

i $\dfrac{|z + 3|}{|z - 6i|} = 1$ j $\dfrac{|z + 6 - i|}{|z - 10 - 5i|} = 1$

**7** Given that $|z - 3| = |z - 6\text{i}|$,

**a** sketch the locus of $z$ **(3 marks)**

**b** find the exact least possible value of $|z|$. **(4 marks)**

**8** Given that $|z + 3 + 3\text{i}| = |z - 9 - 5\text{i}|$,

**a** sketch the locus of $z$ **(3 marks)**

**b** find the Cartesian equation of this locus **(3 marks)**

**c** find the exact least possible value of $|z|$. **(3 marks)**

**9** Sketch the locus of $z$ and give the Cartesian equation of the locus of $z$ when:

**a** $|2 - z| = 3$ **b** $|5\text{i} - z| = 4$ **c** $|3 - 2\text{i} - z| = 3$

**10** Sketch the locus of $z$ when:

**a** $\arg z = \frac{\pi}{3}$ **b** $\arg(z + 3) = \frac{\pi}{4}$ **c** $\arg(z - 2) = \frac{\pi}{2}$

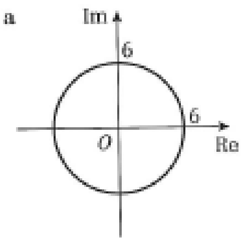
**d** $\arg(z + 2 + 2\text{i}) = -\frac{\pi}{4}$ **e** $\arg(z - 1 - \text{i}) = \frac{3\pi}{4}$ **f** $\arg(z + 3\text{i}) = \pi$

**g** $\arg(z - 1 + 3\text{i}) = \frac{2\pi}{3}$ **h** $\arg(z - 3 + 4\text{i}) = -\frac{\pi}{2}$ **i** $\arg(z - 4\text{i}) = -\frac{3\pi}{4}$

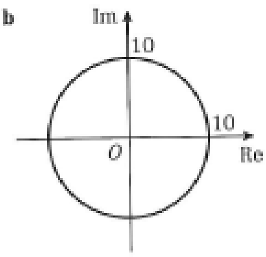
ANSWERS

**Exercise 2E**

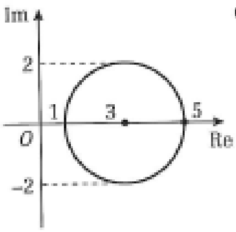
**1 a**



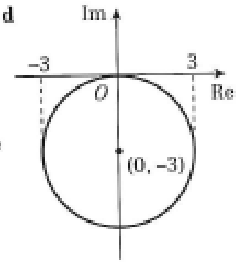

**b**



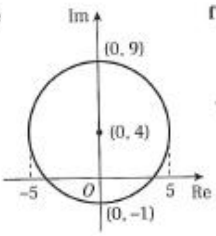

**c**



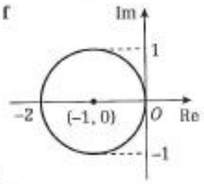

**d**



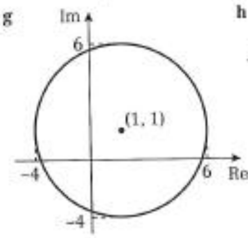

**e**



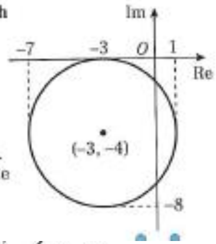

**f**



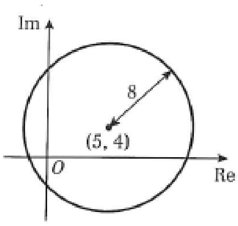

**g**




**h**



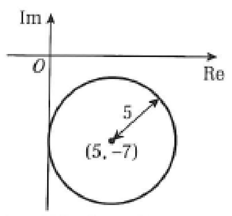

**2 a**




**b i** $(4 + \sqrt{39})\text{i}$ and $(4 - \sqrt{39})\text{i}$

**ii** $5 + 4\sqrt{3}$ and $5 - 4\sqrt{3}$

**3 a**

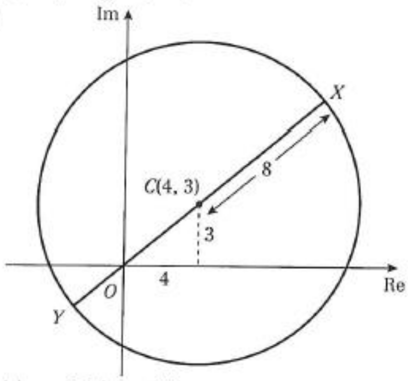



**b** $(x - 5)^2 + (y + 7)^2 = 25$

**c** $2\arctan\left(\frac{5}{7}\right) - \frac{\pi}{2} = -0.330$ rad (3 s.f.)

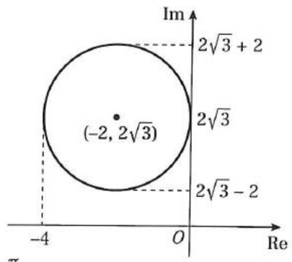
**4 a** $(x - 4)^2 + (y - 3)^2 = 8^2$

**b**




**c** $|z|_{\min} = 3$, $|z|_{\max} = 13$

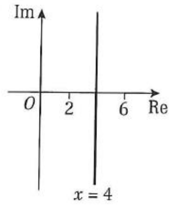
**5 a**



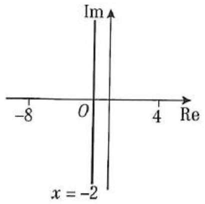

**b** $\frac{\pi}{2}$

**c** 2.51 rad

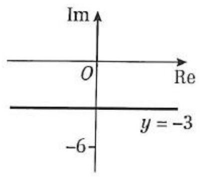
**6 a**



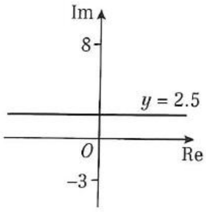

**b**



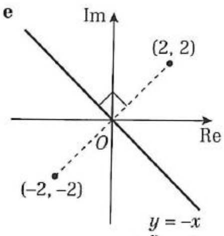

**c**



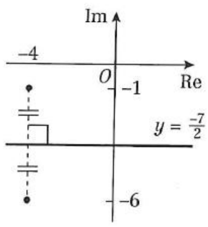

**d**



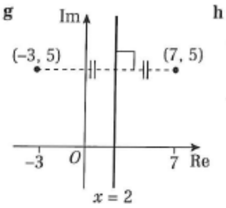

**e**



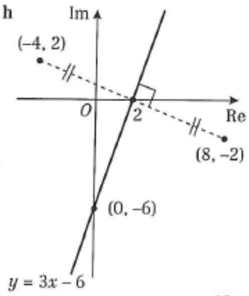

**f**



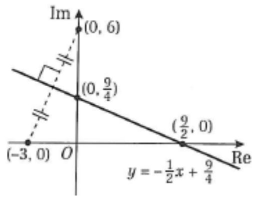

**g**



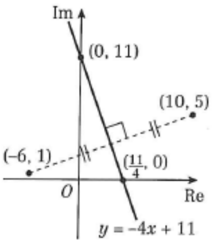

**h**



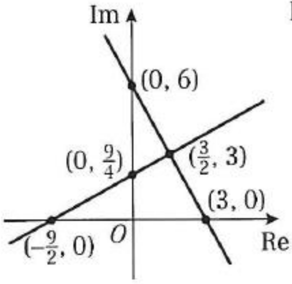

**i**




**j**

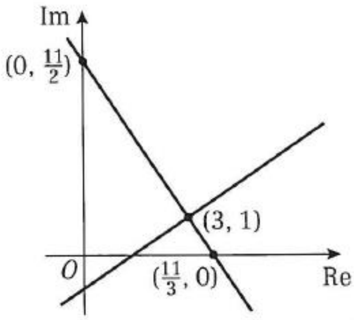



**7 a**




**b** $\frac{9\sqrt{5}}{10}$

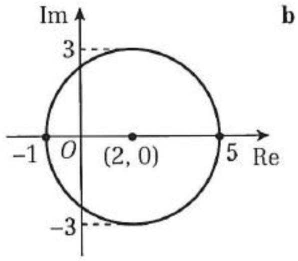
**8 a**



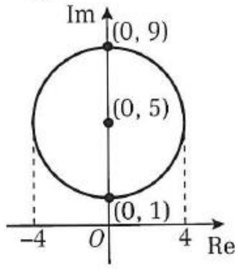

**b** $y = -\frac{3}{2}x + \frac{11}{2}$

**c** $\frac{11\sqrt{13}}{13}$

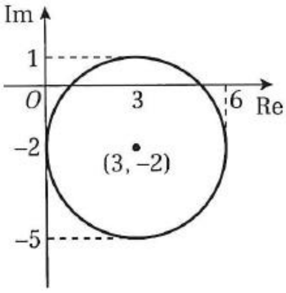
**9 a**



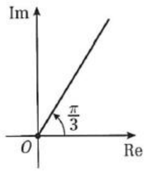

**b**



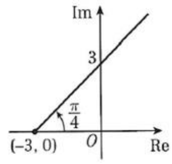

**c**



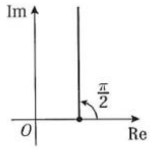

**10 a**



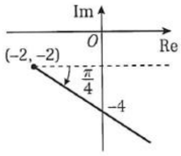

**b**



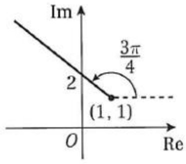

**c**



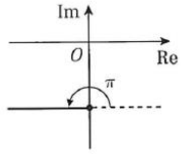

**d**



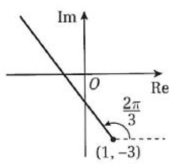

**e**



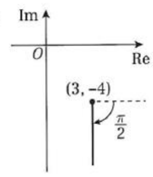

**f**



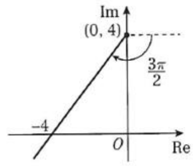

**g**




**h**




**i**



-Edexcel Book Ex2F Q1aceg,2-6

## Exercise 2F

1 On an Argand diagram, shade in the regions represented by the following inequalities:

a $|z| < 3$ b $|z - 2i| > 2$ c $|z + 7| \geqslant |z - 1|$ d $|z + 6| > |z + 2 + 8i|$

e $2 \leqslant |z| \leqslant 3$ f $1 \leqslant |z + 4i| \leqslant 4$ g $3 \leqslant |z - 3 + 5i| \leqslant 5$

2 The region $R$ in an Argand diagram is satisfied by the inequalities $|z| \leqslant 5$ and $|z| \leqslant |z - 6i|$. Draw an Argand diagram and shade in the region $R$. **(6 marks)**

3 The complex number $z$ is represented by a point $P$ on an Argand diagram.
Given that $|z + 1 - i| \leqslant 1$ and $0 \leqslant \arg z \leqslant \frac{3\pi}{4}$, shade the locus of $P$. **(6 marks)**

4 Shade on an Argand diagram the region satisfied by

$$\{z \in \mathbb{C} : |z| \leqslant 3\} \cap \left\{z \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg(z + 3) \leqslant \pi\right\}$$ **(6 marks)**

5 a Sketch on the same Argand diagram:

i the locus of points representing $|z - 2| = |z - 6 - 8i|$ **(2 marks)**

ii the locus of points representing $\arg(z - 4 - 2i) = 0$ **(2 marks)**

iii the locus of points representing $\arg(z - 4 - 2i) = \frac{\pi}{2}$ **(2 marks)**

b Shade on an Argand diagram the set of points

$$\{z \in \mathbb{C} : |z - 2| \leqslant |z - 6 - 8i|\} \cap \left\{z \in \mathbb{C} : 0 \leqslant \arg(z - 4 - 2i) \leqslant \frac{\pi}{2}\right\}$$ **(2 marks)**

6 a Find the Cartesian equations of:

i the locus of points representing $|z + 10| = |z - 6 - 4i\sqrt{2}|$

ii the locus of points representing $|z + 1| = 3$. **(6 marks)**

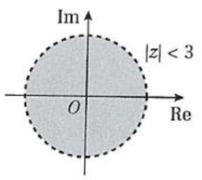
b Find the two values of $z$ that satisfy both $|z + 10| = |z - 6 - 4i\sqrt{2}|$ and $|z + 1| = 3$. **(2 marks)**

c Hence shade in the region $R$ on an Argand diagram which satisfies both $|z + 10| \leqslant |z - 6 - 4i\sqrt{2}|$ and $|z + 1| \leqslant 3$. **(4 marks)**

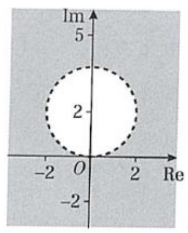
ANSWERS

**Exercise 2F**

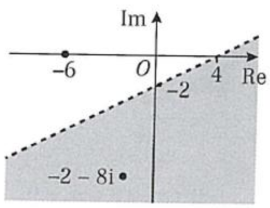
1 a



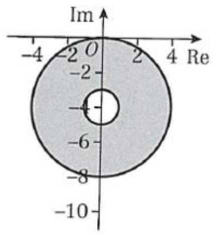

b



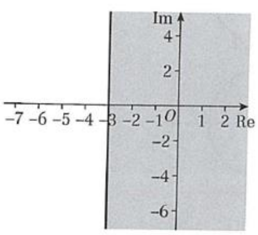

d



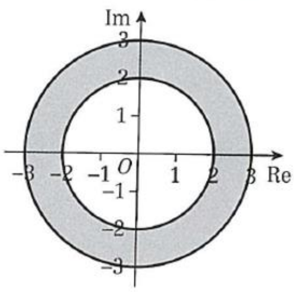

f



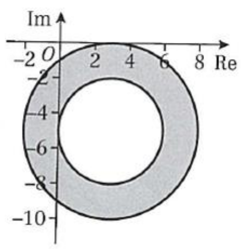

c



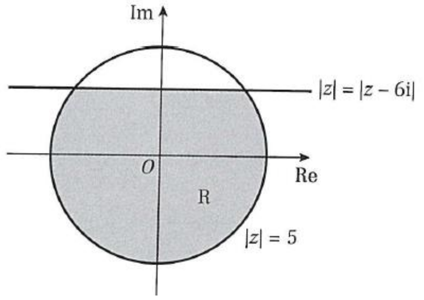

e



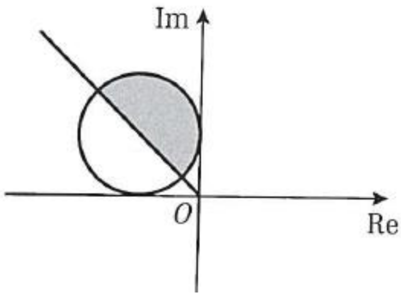

g



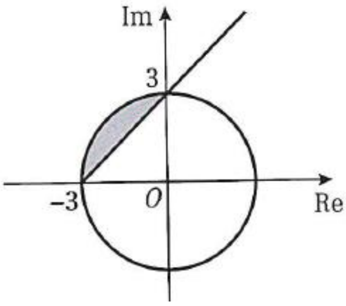
2



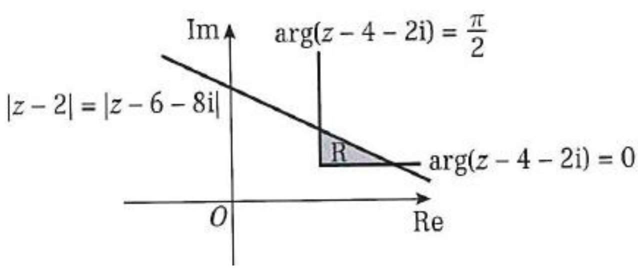

3




4



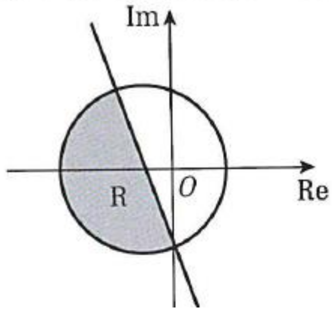

5




6 **a** **i** $y = -2\sqrt{2}x - 2\sqrt{2}$ **ii** $(x + 1)^2 + y^2 = 9$

**b** $z = -\sqrt{2} + 2i\sqrt{2}$ or $z = -2i$

**c**




-Edexcel Book Mixed Exercise 2

## Mixed exercise 2

**1** $f(z) = z^2 + 5z + 10$

**a** Find the roots of the equation $f(z) = 0$, giving your answers in the form $a \pm ib$, where $a$ and $b$ are real numbers. **(3 marks)**

**b** Show these roots on an Argand diagram. **(1 mark)**

**2** $f(z) = z^3 + z^2 + 3z - 5$

Given that $f(-1 + 2i) = 0$,

**a** find all the solutions to the equation $f(z) = 0$ **(4 marks)**

**b** show all the roots of $f(z) = 0$ on a single Argand diagram **(2 marks)**

**c** prove that these three points are the vertices of a right-angled triangle. **(2 marks)**

**3** $f(z) = z^4 - z^3 + 13z^2 - 47z + 34$

Given that $z = -1 + 4i$ is a solution to the equation,

**a** find all the solutions to the equation $f(z) = 0$ **(4 marks)**

**b** show all the roots on a single Argand diagram. **(2 marks)**

4 The real and imaginary parts of the complex number $z = x + iy$ satisfy the equation

$$(4 - 3i)x - (1 + 6i)y - 3 = 0$$

a Find the value of $x$ and the value of $y$. **(3 marks)**

b Show $z$ on an Argand diagram. **(1 mark)**

Find the values of:

c $|z|$ **(2 marks)**

d $\arg z$ **(2 marks)**

5 $z_1 = 4 + 2i$, $z_2 = -3 + i$

a Draw points representing $z_1$ and $z_2$ on the same Argand diagram. **(1 mark)**

b Find the exact value of $|z_1 - z_2|$. **(2 marks)**

Given that $w = \frac{z_1}{z_2}$,

c express $w$ in the form $a + ib$, where $a, b \in \mathbb{R}$ **(2 marks)**

d find $\arg w$, giving your answer in radians. **(2 marks)**

6 A complex number $z$ is given by $z = a + 4i$ where $a$ is a non-zero real number.

a Find $z^2 + 2z$ in the form $x + iy$, where $x$ and $y$ are real expressions in terms of $a$. **(4 marks)**

Given that $z^2 + 2z$ is real,

b find the value of $a$. **(1 mark)**

Using this value for $a$,

c find the values of the modulus and argument of $z$, giving the argument in radians and giving your answers correct to 3 significant figures. **(3 marks)**

d Show the complex numbers $z$, $z^2$ and $z^2 + 2z$ on a single Argand diagram. **(3 marks)**

7 The complex number $z$ is defined by $z = \frac{3 + 5i}{2 - i}$

Find:

a $|z|$ **(4 marks)**

b $\arg z$ **(2 marks)**

8 $z = 1 + 2i$

a Show that $|z^2 - z| = 2\sqrt{5}$. **(4 marks)**

b Find $\arg(z^2 - z)$, giving your answer in radians to 2 decimal places. **(2 marks)**

c Show $z$ and $z^2 - z$ on a single Argand diagram. **(2 marks)**

9 $z = \frac{1}{2 + i}$

a Express in the form $a + bi$, where $a, b \in \mathbb{R}$,

i $z^2$ ii $z - \frac{1}{z}$ **(4 marks)**

b Find $|z^2|$. **(2 marks)**

c Find $\arg\left(z - \frac{1}{z}\right)$, giving your answer in radians to two decimal places. **(2 marks)**

10 $z = \frac{a + 3i}{2 + ai}$, $a \in \mathbb{R}$

a Given that $a = 4$, find $|z|$.

b Show that there is only one value of $a$ for which $\arg z = \frac{\pi}{4}$, and find this value.

**11** $z_1 = -1 - \text{i}$, $z_2 = 1 + \text{i}\sqrt{3}$

**a** Express $z_1$ and $z_2$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$. **(2 marks)**

**b** Find the modulus of:

**i** $z_1 z_2$ **ii** $\frac{z_1}{z_2}$ **(2 marks)**

**c** Find the argument of:

**i** $z_1 z_2$ **ii** $\frac{z_1}{z_2}$ **(2 marks)**

**12** $z = 2 - 2\text{i}\sqrt{3}$

Find:

**a** $|z|$ **(1 mark)**

**b** $\arg z$, in terms of $\pi$. **(2 marks)**

$w = 4\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)$

Find:

**c** $\left|\frac{w}{z}\right|$ **(1 mark)**

**d** $\arg\left(\frac{w}{z}\right)$, in terms of $\pi$. **(2 marks)**

**13** Express $4 - 4\text{i}$ in the form $r(\cos\theta + \text{i}\sin\theta)$, where $r > 0$, $-\pi < \theta \leqslant \pi$, giving $r$ and $\theta$ as exact values. **(3 marks)**

**14** The point $P$ represents a complex number $z$ in an Argand diagram.
Given that $|z + 1 - \text{i}| = 1$,

**a** find a Cartesian equation for the locus of $P$ **(2 marks)**

**b** sketch the locus of $P$ on an Argand diagram **(2 marks)**

**c** find the greatest and least possible values of $|z|$ **(2 marks)**

**d** find the greatest and least possible values of $|z - 1|$. **(2 marks)**

**15** Given that $\arg(z - 2 + 4\text{i}) = \frac{\pi}{4}$,

**a** sketch the locus of $P(x, y)$ which represents $z$ on an Argand diagram

**b** find the minimum value of $|z|$ for points on this locus.

**16** The complex number $z$ satisfies $|z + 3 - 6\text{i}| = 3$. Show that the exact maximum value of $\arg z$ in the interval $(-\pi, \pi)$ is $\frac{\pi}{2} + 2\arcsin\left(\frac{1}{\sqrt{5}}\right)$. **(4 marks)**

**17** A complex number $z$ is represented by the point $P$ on the Argand diagram.
Given that $|z - 5| = 4$,

**a** sketch the locus of $P$. **(2 marks)**

**b** Find the complex numbers that satisfy both $|z - 5| = 4$ and $\arg(z + 3\text{i}) = \frac{\pi}{3}$, giving your answers in radians to 2 decimal places. **(6 marks)**

**c** Given that $\arg(z + 5) = \theta$ and $|z - 5| = 4$ have no common solutions, find the range of possible values of $\theta$, $-\pi < \theta < \pi$. **(3 marks)**

**18** Given that $|z + 5 - 5\mathrm{i}| = |z - 6 - 3\mathrm{i}|$,

**a** sketch the locus of $z$ **(3 marks)**

**b** find the Cartesian equation of this locus **(3 marks)**

**c** find the least possible value of $|z|$. **(3 marks)**

**19 a** Find the Cartesian equation of the locus of points that satisfies $|z - 4| = |z - 8\mathrm{i}|$. **(3 marks)**

**b** Find the value of $z$ that satisfies both $|z - 2| = |z - 4\mathrm{i}|$ and $\arg z = \frac{\pi}{4}$ **(3 marks)**

**c** Shade on an Argand diagram the set of points

$\{z \in \mathbb{C} : |z - 4| \leqslant |z - 8\mathrm{i}|\} \cap \left\{z \in \mathbb{C} : \frac{\pi}{4} \leqslant \arg z \leqslant \pi\right\}$ **(3 marks)**

**20 a** Find the Cartesian equations of:

**i** the locus of points representing $|z - 3 + \mathrm{i}| = |z - 1 - \mathrm{i}|$

**ii** the locus of points representing $|z - 2| = 2\sqrt{2}$. **(6 marks)**

**b** Find the two values of $z$ that satisfy both $|z - 3 + \mathrm{i}| = |z - 1 - \mathrm{i}|$ and $|z - 2| = 2\sqrt{2}$. **(2 marks)**

The region $R$ is defined by the inequalities $|z - 3 + \mathrm{i}| \geqslant |z - 1 - \mathrm{i}|$ and $|z - 2| \leqslant 2\sqrt{2}$.

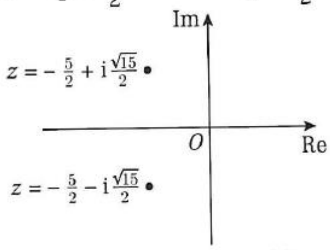
**c** Show the region $R$ on an Argand diagram. **(4 marks)**

ANSWERS

**Mixed exercise 2**

**1 a** $z = -\frac{5}{2} + \frac{\sqrt{15}}{2}\mathrm{i}$ and $z = -\frac{5}{2} - \frac{\sqrt{15}}{2}\mathrm{i}$

**b**




**2 a** $-1 + 2\mathrm{i}$, $-1 - 2\mathrm{i}$ are two of the roots. These roots can be used to form the quadratic $z^2 + 2z + 5$.
$(z - 1)(z^2 + 2z + 5) = \mathrm{f}(z)$, so third root is 1.

**b** Argand diagram showing $-1 + 2\mathrm{i}$, $-1 - 2\mathrm{i}$ and 1.

**c** Sides of triangle are $\sqrt{8}$, $\sqrt{8}$ and 4. $(\sqrt{8})^2 + (\sqrt{8})^2 = 4^2$.

**3 a** $-1 + 4\mathrm{i}$, $-1 - 4\mathrm{i}$, 2, 1

**b** Argand diagram showing above roots.

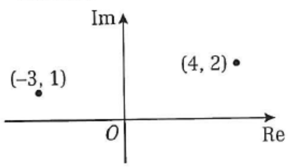
**4 a** $4x - y = 3$
$-3x - 6y = 0 \Rightarrow x = -2y$
$-9y = 3 \Rightarrow y = -\frac{1}{3} \Rightarrow x = \frac{2}{3}$

**b** Argand diagram showing the point $z = \frac{2}{3} - \frac{1}{3}i$

**c** $\frac{\sqrt{5}}{3}$

**d** $-0.46$ rad

**5 a**




**b** $5\sqrt{2}$ **c** $-1 - \mathrm{i}$ **d** $-\frac{3\pi}{4}$

**6 a** $z^2 = (a^2 - 16) + 8a\mathrm{i}$
$2z = 2a + 8\mathrm{i}$
$z^2 + 2z = (a^2 + 2a - 16) + (8 + 8a)\mathrm{i}$

**b** $a = -1$

**c** $z = -1 + 4\mathrm{i}$
$|z| = \sqrt{17} \approx 4.12$
$\arg z \approx 1.82$

**d** Show $z = -1 + 4\mathrm{i}$, $z^2 = -15 - 8\mathrm{i}$ and $z^2 + 2z = -17$ on a single Argand diagram.

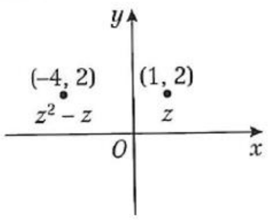
**7 a** $z = \frac{(3 + 5\mathrm{i})(2 + \mathrm{i})}{(2 - \mathrm{i})(2 + \mathrm{i})} = \frac{1}{5} + \frac{13}{5}\mathrm{i}$
$|z| = \frac{1}{5}\sqrt{170}$

**b** $\arg z = 1.49$

**8 a** $z^2 = -3 + 4\mathrm{i}$
$z^2 - z = -4 + 2\mathrm{i}$
$|-4 + 2\mathrm{i}| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}$

**b** 2.68

**c**




**9 a i** $\frac{3}{25} - \frac{4}{25}\mathrm{i}$ **ii** $\frac{-8}{5} - \frac{6}{5}\mathrm{i}$ **b** $\frac{1}{5}$ **c** $-2.50$

**10 a** $\frac{\sqrt{5}}{2}$

**b** $\frac{a + 3\mathrm{i}}{2 + a\mathrm{i}} = \frac{5a}{4 + a^2} + \frac{-a^2 + 6}{4 + a^2}\mathrm{i}$,
for $\arg z = \frac{\pi}{4}$ real and imaginary parts must be equal
$\Rightarrow a^2 + 5a - 6 = 0$
$\Rightarrow a = -6$ or 1
$a$ cannot be negative otherwise $\arg z$ is negative
$\therefore a = 1$

**11 a** $z_1 = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$ and
$z_2 = 2\left(\cos\left(\frac{\pi}{3}\right) + \mathrm{i}\sin\left(\frac{\pi}{3}\right)\right)$

**b i** $2\sqrt{2}$ **ii** $\frac{\sqrt{2}}{2}$

**c i** $-\frac{5\pi}{12}$ **ii** $\frac{11\pi}{12}$

**12 a** $|z| = |2 - 2\mathrm{i}\sqrt{3}| = \sqrt{2^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$

**b** $\arg z = -\frac{\pi}{3}$

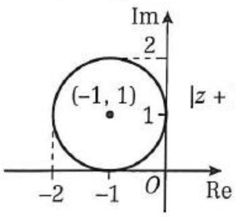
**c** $\left|\frac{w}{z}\right| = 1$

**d** $\arg\left(\frac{w}{z}\right) = \frac{\pi}{12}$

**13** $4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$

**14 a** $(x+1)^2 + (y-1)^2 = 1$

**b**



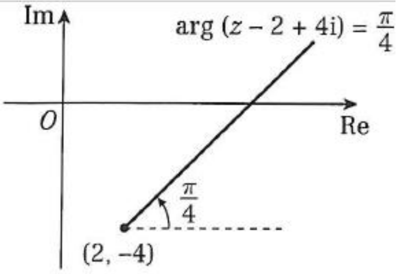

**c** $|z|_{\min} = \sqrt{2} - 1$

$|z|_{\max} = \sqrt{2} + 1$

**d** $|z-1|_{\min} = \sqrt{5} - 1$

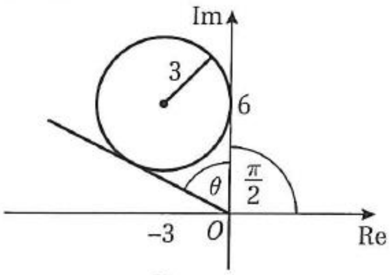
$|z-1|_{\max} = \sqrt{5} + 1$

**15 a**




**b** $3\sqrt{2}$

**16**

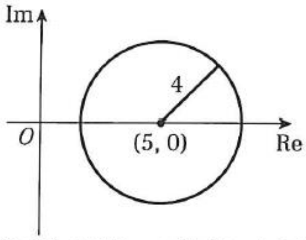



Max value $= \frac{\pi}{2} + \theta$

$$\sin\left(\frac{\theta}{2}\right) = \frac{3}{\sqrt{3^2+6^2}} = \frac{3}{\sqrt{45}} = \frac{1}{\sqrt{5}}$$

$$\Rightarrow \frac{\pi}{2} + \theta = \frac{\pi}{2} + 2\arcsin\left(\frac{1}{\sqrt{5}}\right)$$

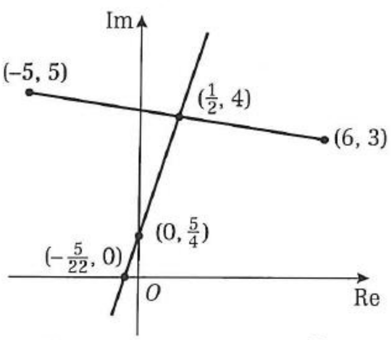
**17 a**




**b** (3.96, 3.86) and (1.14, −1.03)

**c** $-\pi < \theta < -0.41,\ 0.41 < \theta < \pi$

**18 a**



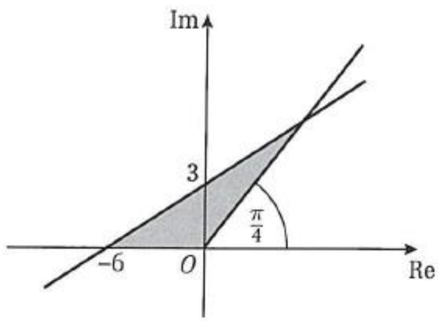

**b** $y = \frac{11}{2}x + \frac{5}{4}$

**c** $\frac{\sqrt{5}}{10}$

**19 a** $y = \frac{1}{2}x + 3$

**b** $6 + 6\mathrm{i}$

**c**




**20 a i** $y = x - 2$

**ii** $(x-2)^2 + y^2 = 8$

**b** $-2\mathrm{i},\ 4 + 2\mathrm{i}$

**c**