Vectors Questions

*UPM Ex2B Q3,4,6,7,9,10,11,13,18,19,20

3. If $a\mathbf{i} + 8\mathbf{j}$ is parallel to $2\mathbf{i} + 4\mathbf{j}$, find the value of a.

4. If $\mathbf{a} = \mathbf{i} + 2\mathbf{j}$ find $\hat{\mathbf{a}}$, a unit vector in the direction of $\mathbf{a}$.

5. If $\mathbf{b} = 3\mathbf{i} - \mathbf{j}$ find $\hat{\mathbf{b}}$, a unit vector in the direction of $\mathbf{b}$.

6. Find a vector that is of magnitude 39 units and is parallel to $5\mathbf{i} + 12\mathbf{j}$.

7. Find a vector that is of magnitude $3\sqrt{5}$ units and is parallel to $2\mathbf{i} - \mathbf{j}$.

8. Find a vector that is of magnitude 2 units and is parallel to $4\mathbf{i} - 3\mathbf{j}$.

9. If the point P has position vector $2\mathbf{i} + 3\mathbf{j}$ and point Q has position vector $7\mathbf{i} + 4\mathbf{j}$, find: (a) $\overrightarrow{PQ}$, (b) $\overrightarrow{QP}$.

10. If the point P has position vector $7\mathbf{i} - 3\mathbf{j}$ and point Q has position vector $5\mathbf{i} + 5\mathbf{j}$, find: (a) $\overrightarrow{PQ}$, (b) $\overrightarrow{QP}$.

11. The point P has position vector $-5\mathbf{i} + 3\mathbf{j}$ and Q is a point such that $\overrightarrow{PQ} = 7\mathbf{i} - \mathbf{j}$. Find the position vector of Q.

12. The point P has position vector $3\mathbf{i} - 2\mathbf{j}$ and Q is a point such that $\overrightarrow{QP} = 2\mathbf{i} - 3\mathbf{j}$. Find the position vector of Q.

13. Using $\mathbf{a} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ as base vectors, express the vectors

$\mathbf{c} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ and $\mathbf{d} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ in the form $\lambda\mathbf{a} + \mu\mathbf{b}$.

18. The point K has position vector $3\mathbf{i} + 2\mathbf{j}$ and point L has position vector $\mathbf{i} + 3\mathbf{j}$. Find the position vector of the point which divides KL in the ratio (a) $4:3$, (b) $4:-3$.

19. The three points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. If $\mathbf{c} = 3\mathbf{b} - 2\mathbf{a}$, show that A, B and C are collinear.

20. The three points A, B and C have position vectors $\mathbf{i} - \mathbf{j}$, $5\mathbf{i} - 3\mathbf{j}$ and $11\mathbf{i} - 6\mathbf{j}$ respectively. Show that A, B and C are collinear.

ANSWERS Ex2B

1. $\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OB} = 3\mathbf{i}$, $\overrightarrow{OC} = -\mathbf{i} + 3\mathbf{j}$, $\overrightarrow{OD} = -3\mathbf{i} + 2\mathbf{j}$, $\overrightarrow{OE} = -2\mathbf{i} - \mathbf{j}$, $\overrightarrow{OF} = -2\mathbf{i} - 2\mathbf{j}$, $\overrightarrow{OG} = -2\mathbf{j}$, $\overrightarrow{OH} = 3\mathbf{i} - 3\mathbf{j}$

2. (a) (i) $4\mathbf{i} + 3\mathbf{j}$ (ii) 5 units (iii) 36·8° (b) (i) $-4\mathbf{i} + \mathbf{j}$ (ii) 4·12 units (iii) 14° (c) (i) $2\mathbf{i} - 3\mathbf{j}$ (ii) 3·61 units (iii) 56·3°

3. 4 **4.** $\frac{\sqrt{5}}{5}(\mathbf{i} + 2\mathbf{j})$ **5.** $\frac{\sqrt{10}}{10}(3\mathbf{i} - \mathbf{j})$ **6.** $15\mathbf{i} + 36\mathbf{j}$ **7.** $6\mathbf{i} - 3\mathbf{j}$

8. $\frac{2}{5}(4\mathbf{i} - 3\mathbf{j})$ **9.** (a) $5\mathbf{i} + \mathbf{j}$ (b) $-5\mathbf{i} - \mathbf{j}$ **10.** (a) $-2\mathbf{i} + 8\mathbf{j}$ (b) $2\mathbf{i} - 8\mathbf{j}$

11. $2\mathbf{i} + 2\mathbf{j}$ **12.** $\mathbf{i} + \mathbf{j}$

13. $\mathbf{a} + 2\mathbf{b}$, $\frac{3}{11}\mathbf{a} + \frac{23}{11}\mathbf{b}$ **14.** (a) $\sqrt{10}$ (b) $\sqrt{5}$ (c) $5\mathbf{i}$ (d) 5

15. (a) $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ (b) 5 (c) $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$ (d) $\sqrt{58}$ **16.** $6\mathbf{i} + \mathbf{j}$ **17.** $\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

18. (a) $1\frac{6}{7}\mathbf{i} + 2\frac{4}{7}\mathbf{j}$ (b) $-5\mathbf{i} + 6\mathbf{j}$ **21.** C, E and F

*UPM Ex2C Q1,3,5,6,8,11,12,15,16

1. If **a** = **i** + 2**j**, **b** = **i** − 2**j**, **c** = 2**i** − 3**j** and **d** = 6**i** + 3**j**, find which two of these vectors are perpendicular to each other.
2. If **e** = −**i** − 3**j**, **f** = **i** + 3**j**, **g** = −3**i** − 2**j** and **h** = 6**i** − 9**j**, find which two of these vectors are perpendicular to each other.
3. Find the angle between the vectors **a** and **b** given that **a** = 3**i** + 4**j** and **b** = 5**i** + 12**j**. (Give your answer to the nearest degree.)
4. Find the angle between the vectors **c** and **d** given that **c** = 5**i** − **j** and **d** = 2**i** + 3**j**. (Give your answer to the nearest degree.)
5. Find the angle between the vectors **e** and **f** given that **e** = −**i** − 2**j** and **f** = 2**i** + **j**. (Give your answer to the nearest degree.)
6. If the angle between the vectors **c** = a**i** + 2**j** and **d** = 3**i** + **j** is 45° find the two possible values of a.

8. The points A, B, C and D have position vectors 5**i** + **j**, −3**i** + 2**j**, −3**i** − 3**j** and **i** −6**j** respectively. Show that AC is perpendicular to BD.
9. The points E, F and G have position vectors 2**i** + 2**j**, **i** + 6**j** and −7**i** + 4**j**. Show that the triangle EFG is right-angled at F.
10. The points A, B, C and D have position vectors **a**, **b**, **c** and **d** respectively where **a** = −2**j**, **b** = −2**i** + 4**j**, **c** = 3**i** + 4**j** and **d** = 4**i** + y**j**. If AC is perpendicular to BD, find the value of y.
11. The points A, B, C and D have position vectors −2**i** + **j**, 7**j**, 3**i** + 6**j** and x**i** + y**j** respectively. If $|\overrightarrow{AC}| = |\overrightarrow{BD}|$ and AC is perpendicular to BD, find the two possible values of x and the corresponding values of y.

12. If **a** = $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ find: (a) a unit vector parallel to **a**,
(b) a unit vector perpendicular to **a**.

15. If $\overrightarrow{OA} = 2\mathbf{a} + 3\mathbf{b}$ and $\overrightarrow{OB} = 3\mathbf{a} - 2\mathbf{b}$ show that $\overrightarrow{OA} \cdot \overrightarrow{OB} = 6a^2 + 5\mathbf{a} \cdot \mathbf{b} - 6b^2$.
16. If $\overrightarrow{OC} = 2\mathbf{a} + 3\mathbf{b}$ and $\overrightarrow{OD} = 2\mathbf{a} - 3\mathbf{b}$ show that $\overrightarrow{OC} \cdot \overrightarrow{OD} = 4a^2 - 9b^2$.

ANSWERS Ex2C

1. **b** and **d** **2.** **g** and **h** **3.** 14° **4.** 68° **5.** 143° **6.** −4 or 1
7. (a) $x^2 + y^2 = 25$ (b) $2x + y = 0$ (c) $x = 2y$ (d) $4x + y = 9$ **10.** 1
11. $x = 5, y = 2; x = -5, y = 12$ **12.** (a) $\begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ (b) $\pm\begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}$ **13.** (a) $\begin{pmatrix} 7\sqrt{2} \\ -\sqrt{2} \end{pmatrix}$ (b) $\pm\begin{pmatrix} 3 \\ 21 \end{pmatrix}$
14. (a) −3**i** + 4**j** (b) −6**i** − **j** (c) 9**i** − 3**j** (d) 35°, 117°, 28° **17.** (b) 23 **18.** 131°
19. 4 **20.** [illegible]$\sqrt{13}$ **21.** [illegible]$\sqrt{5}$ **22.** (b) $k\sqrt{7}, 2k\sqrt{21}$

*UPM Ex17A Q1,2,4,6,7,11,12

1. If $\mathbf{a} = 9\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ find
   (a) $|\mathbf{a}|$ (b) $|\mathbf{b}|$ (c) $|\mathbf{c}|$ (d) $\mathbf{a} \cdot \mathbf{b}$ (e) $\mathbf{b} \cdot \mathbf{c}$
   (f) the angle between **a** and **b** (to the nearest degree)
   (g) the angle between **b** and **c** (to the nearest degree).

2. If $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ find
   (a) $|\mathbf{a}|$ (b) $|\mathbf{b}|$ (c) the angle between **a** and **b** (to the nearest degree).

3. If $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$ find $\hat{\mathbf{a}}$, a unit vector in the direction of **a**.

4. If $\mathbf{b} = \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$ find $\hat{\mathbf{b}}$, a unit vector in the direction of **b**.

5. Find a vector that is perpendicular to $5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

6. Find a unit vector that is perpendicular to $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$.

7. State which of the vectors **a**, **b**, **c** or **d** listed below are perpendicular to the vector $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.
   $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ $\quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
   $\mathbf{c} = 3\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ $\quad \mathbf{d} = 4\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

11. Points A, B and C have position vectors $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ and $5\mathbf{i} + 12\mathbf{j} - 7\mathbf{k}$ respectively. Prove that A, B and C are collinear.

12. Points A, B and C have position vectors $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, $3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} - \mathbf{k}$ respectively. Prove that angle $B\hat{A}C$ is a right angle.

ANSWERS Ex17A

1. (a) 11 (b) 7 (c) 3 (d) 12 (e) 16 (f) 81° (g) 40°

2. (a) $\sqrt{11}$ (b) $\sqrt{6}$ (c) 76° 3. $\frac{3}{13}\mathbf{i} + \frac{4}{13}\mathbf{j} + \frac{12}{13}\mathbf{k}$ 4. $\begin{pmatrix} \frac{4}{9} \\ \frac{4}{9} \\ -\frac{7}{9} \end{pmatrix}$

5. $\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is one example 6. $\frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$ is one example

7. **c** and **d** 8. $14 : -2 : 5$, $\frac{14}{15}, -\frac{2}{15}, \frac{1}{3}$ 9. $1 : 2 : -2$

10. (a) $2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ or any multiple thereof (b) $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$ (c) $\frac{1}{7}(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

15. 36°, 68°, 76° 17. $4\mathbf{a} + \mathbf{b} - 2\mathbf{c}$ 18. $3\mathbf{a} - \mathbf{b} + 2\mathbf{c}$

*UPM Ex17C Q1-4,6,7,9a,10-13,15-17,19,20ac

1. State the vector equation of the line which is parallel to $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and which passes through the point A, position vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.
2. State the vector equation of the line which passes through the point B, position vector $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ and which is parallel to the vector $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.
3. State a vector that is parallel to the line with vector equation $\mathbf{r} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k})$.
4. Show that the point with position vector $4\mathbf{i} - \mathbf{j} + 12\mathbf{k}$ lies on the line with vector equation $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$.
5. Points A, B and C have position vectors $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 1 \\ 7 \end{pmatrix}$ respectively. Find which of these points lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.
6. Points D, E and F have position vectors $\mathbf{i} - 2\mathbf{j}$, $4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $7\mathbf{i} - 8\mathbf{j} - 4\mathbf{k}$ respectively. Find which of these points lie on the line with vector equation $\mathbf{r} = (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} - \mathbf{k})$.
7. If the point A, position vector $a\mathbf{i} + b\mathbf{j} + 3\mathbf{k}$, lies on the line L, vector equation $\mathbf{r} = (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) + \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k})$, find the values of a and b.

9. Find the cartesian equations of the lines with vector equations
   (a) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + \mathbf{k})$,
   (b) $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$,
   (c) $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} + \eta(2\mathbf{i} - \mathbf{j} - \mathbf{k})$.
10. Find the vector equation of the line with parametric equations $x = 2 + 3\lambda$, $y = 5 - 2\lambda$, $z = 4 - \lambda$
11. Find the vector equations of the lines with the following cartesian equations
   (a) $\dfrac{x-2}{3} = \dfrac{y-2}{2} = \dfrac{z+1}{4}$ (b) $x - 3 = \dfrac{y+2}{4} = \dfrac{z-3}{-1}$
12. Lines L_1 and L_2 have vector equations $\mathbf{r} = 8\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(-4\mathbf{i} + \mathbf{j})$ and $\mathbf{r} = -2\mathbf{i} + 8\mathbf{j} - \mathbf{k} + \mu(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$ respectively. Show that L_1 and L_2 intersect and find the position vector of the point of intersection.

13. Lines L_1 and L_2 have vector equations

$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 8 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ respectively. Show that L_1 and L_2 intersect and find the position vector of the point of intersection.

15. For each of the pairs of lines given by the following vector equations state whether the lines are parallel lines, non-parallel coplanar lines or skew lines.

(a) $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ and $\mathbf{r} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k} + \mu(3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k})$.

(b) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 7\mathbf{i} + 3\mathbf{j} + 5\mathbf{k} + \mu(-\mathbf{i} + 2\mathbf{j})$.

(c) $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} + 7\mathbf{j} + 6\mathbf{k} + \mu(3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$.

(d) $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = -8\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \mu(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$.

16. Find the acute angle between the lines with vector equations $\mathbf{r} = 2\mathbf{i} + \mathbf{j} - \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$ and $\mathbf{r} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} + \mu(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, giving your answer to the nearest degree.

17. Find the acute angle between the lines whose equations are $\frac{x-2}{-4} = \frac{y-3}{3} = \frac{z+1}{-1}$ and $\frac{x-3}{2} = \frac{y-1}{6} = \frac{z+1}{-5}$, giving your answer to the nearest degree.

18. The vector equations of three lines are:

line 1 $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

line 2 $\mathbf{r} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k} + \mu(-\mathbf{i} - 2\mathbf{k})$

line 3 $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + \eta(2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

(a) Show that lines 1 and 2 intersect and find the position vector of the point of intersection.

(b) Show that lines 2 and 3 intersect and find the position vector of the point of intersection.

(c) Find the distance between these two points of intersection.

19. Two lines L_1 and L_2 lie in the x–y plane and have cartesian equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Show that the vector equations of L_1 and L_2 can be written

$$\mathbf{r}_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ m_1 \end{pmatrix} \quad \text{and} \quad \mathbf{r}_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ m_2 \end{pmatrix}.$$

Use vector methods to show that if θ is the angle between L_1 and L_2

then $\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$.

(i.e. obtain the result of page 380 by vector methods).

20. For each of the following parts find the perpendicular distance from the given point to the given line,

(a) the point with position vector $\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ and the line $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) the point with position vector $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and the line $\mathbf{r} = \mathbf{i} - 6\mathbf{j} - 2\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

(c) the point $(1, 1, 3)$ and the line $\frac{x+4}{2} = \frac{y+1}{3} = \frac{z-1}{3}$

(d) the point $(-6, -4, -5)$ and the line $x - 5 = \frac{y-6}{2} = \frac{z-3}{4}$

ANSWERS

page 420 **Exercise 17C**

1. $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ **2.** $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

3. $2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$ or multiples thereof **5.** B and C **6.** F **7.** 6, 8

8. (a) $3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ (b) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + \lambda(3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k})$

9. (a) $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z+1}{1}$ (b) $\frac{x-3}{3} = \frac{y+1}{2} = \frac{z-2}{-4}$ (c) $\frac{x-2}{2} = \frac{y-1}{-1} = \frac{z-1}{-1}$

10. $\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} - \mathbf{k})$

11. (a) $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$ (b) $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} - \mathbf{k})$

12. $-4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ **13.** $\begin{pmatrix} 9 \\ 3 \\ 0 \end{pmatrix}$

14. (a) (5, 0, 1) (b) lines do not intersect (c) (4, 5, 9) (d) (12, −3, 3)

15. (a) parallel (b) non parallel coplanar (c) skew (d) non parallel coplanar

16. 79° **17.** 69° **18.** (a) $\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$ (b) $-4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$ (c) $5\sqrt{5}$ units

20. (a) $\sqrt{5}$ units (b) $3\sqrt{2}$ units (c) $\sqrt{11}$ units (d) $4\sqrt{6}$ units

21. (a) $\frac{1}{3}\sqrt{21}$ units (b) $\frac{1}{2}\sqrt{14}$ units

22. $\frac{2}{3}\sqrt{35}$ units **23.** (a) $\sqrt{3}$ units (b) $\frac{4}{7}\sqrt{21}$ units

*P4 Book page 213 EX6A Q1alt,2,4,5,6,7alt,9,10

1 Simplify as much as possible:

(a) $3\mathbf{i} \times \mathbf{j}$

(b) $2\mathbf{i} \times (\mathbf{i}+\mathbf{j}+\mathbf{k})$

(c) $(\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k}) + (\mathbf{k}+\mathbf{i}) \times (\mathbf{j}-\mathbf{k})$

(d) $(\mathbf{i}+2\mathbf{j}-\mathbf{k}) \times (2\mathbf{i}-\mathbf{j}+\mathbf{k})$

(e) $(2\mathbf{i}-3\mathbf{j}+\mathbf{k}) \times (-\mathbf{i}+2\mathbf{j}-4\mathbf{k})$

(f) $(\mathbf{i}-\mathbf{j}+\mathbf{k}) \times (3\mathbf{i}-3\mathbf{j}+3\mathbf{k})$

(g) $(2\mathbf{i}+\mathbf{j}-2\mathbf{k}) \times (-3\mathbf{i}+4\mathbf{k})$

(h) $(2\mathbf{i}+\mathbf{k}) \times (\mathbf{i}-2\mathbf{j}+3\mathbf{k})$

(i) $(2\mathbf{i}+3\mathbf{j}-\mathbf{k}) \times (2\mathbf{i}-\mathbf{j}+3\mathbf{k})$

(j) $(-\mathbf{i}+2\mathbf{j}-3\mathbf{k}) \times (5\mathbf{i}-4\mathbf{k})$

2 Find a unit vector which is perpendicular to the vector $(4\mathbf{i}+4\mathbf{j}-7\mathbf{k})$ and to the vector $(2\mathbf{i}+2\mathbf{j}+\mathbf{k})$. [L]

3 Find a unit vector perpendicular to both $2\mathbf{i}-6\mathbf{j}-3\mathbf{k}$ and $4\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

4 Find a vector of magnitude 7 which is perpendicular to both $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$ and $\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

5 Find the magnitude of the vector $(\mathbf{i}+\mathbf{j}-\mathbf{k}) \times (\mathbf{i}-\mathbf{j}+\mathbf{k})$. [L]

6 Given that $\mathbf{a} = -\mathbf{i}+2\mathbf{j}-5\mathbf{k}$ and $\mathbf{b} = 5\mathbf{i}-2\mathbf{j}+\mathbf{k}$ find

(a) $\mathbf{a}.\mathbf{b}$

(b) $\mathbf{a} \times \mathbf{b}$

(c) the unit vector in the direction of $\mathbf{a} \times \mathbf{b}$. [L]

7 Find the sine of the angle between $\mathbf{a}$ and $\mathbf{b}$ where

(a) $\mathbf{a} = 2\mathbf{i}-\mathbf{j}$, $\mathbf{b} = \mathbf{i}+\mathbf{j}-\mathbf{k}$

(b) $\mathbf{a} = \mathbf{i}+\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = -\mathbf{i}+3\mathbf{k}$

(c) $\mathbf{a} = -2\mathbf{i}+\mathbf{j}+\mathbf{k}$, $\mathbf{b} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$

(d) $\mathbf{a} = \mathbf{i}-2\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$

(e) $\mathbf{a} = -\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+3\mathbf{j}-\mathbf{k}$

8 Given that $\mathbf{a} = \mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $\mathbf{b} = p\mathbf{j}+q\mathbf{k}$ and that $\mathbf{a} \times \mathbf{b} = 2\mathbf{j}+\lambda\mathbf{k}$, find the values of the scalar constants p, q and λ.

9 Given that $\mathbf{u} = 2\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $\mathbf{v} = a\mathbf{i}+b\mathbf{k}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{i}+c\mathbf{k}$, find the values of the scalar constants a, b and c. Find, in surd form, the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

10 Given that $\mathbf{r} = a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, $\mathbf{k} \times \mathbf{r} = \mathbf{p}$, $\mathbf{r} \times \mathbf{p} = \mathbf{k}$, where a, b, c are scalar constants, show that

$$a^2+b^2=1 \quad \text{and} \quad c=0$$

ANSWERS

Exercise 6A

1 (a) $3\mathbf{k}$ (b) $-2\mathbf{j}+2\mathbf{k}$

(c) $2\mathbf{k}$ (d) $\mathbf{i}-3\mathbf{j}-5\mathbf{k}$

(e) $10\mathbf{i}+7\mathbf{j}+\mathbf{k}$ (f) $\mathbf{0}$

(g) $4\mathbf{i}-2\mathbf{j}+3\mathbf{k}$ (h) $2\mathbf{i}-5\mathbf{j}-4\mathbf{k}$

(i) $8\mathbf{i}-8\mathbf{j}-8\mathbf{k}$ (j) $-8\mathbf{i}-19\mathbf{j}-10\mathbf{k}$

2 $\frac{1}{\sqrt{2}}(\mathbf{i}-\mathbf{j})$ **3** $\frac{1}{7}(3\mathbf{i}-2\mathbf{j}+6\mathbf{k})$

4 $\frac{7}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$ **5** $2\sqrt{2}$

6 (a) -14 (b) $-8(\mathbf{i}+3\mathbf{j}+\mathbf{k})$

(c) $-\frac{1}{\sqrt{11}}(\mathbf{i}+3\mathbf{j}+\mathbf{k})$

7 (a) $\sqrt{\frac{14}{15}}$ (b) $\sqrt{\frac{23}{55}}$ (c) $\frac{5\sqrt{3}}{9}$

(d) $\frac{3\sqrt{3}}{14}$ (e) $\frac{\sqrt{7}}{14}$

8 $p=2,\quad q=-2,\quad \lambda=2$

9 $a=-1,\quad b=-1,\quad c=-1;\quad -\frac{2\sqrt{2}}{3}$

*P4 book page 220 Ex6B Q1,3,5-11,13,14,16

1 Find the area of the triangle with vertices $A(0, 0, 0)$, $B(1, 2, 1)$ and $C(-1, 3, 3)$.

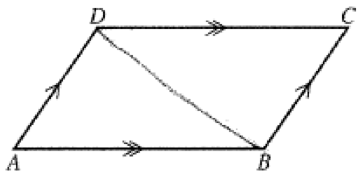
2 Find the area of the triangle with vertices $A(-5, 1, 4)$, $B(0, 0, 0)$ and $C(-2, 3, -1)$.

3 Find the area of the triangle with vertices $A(1, -2, 3)$, $B(-1, -1, 4)$ and $C(-2, 1, 5)$.

4 Find the area of the triangle with vertices $A(2, -1, -1)$, $B(-2, 1, -3)$ and $C(1, -1, 0)$.

5 Find the area of the triangle with vertices $A(-1, 3, 1)$, $B(2, 2, -3)$ and $C(-1, 3, -4)$.

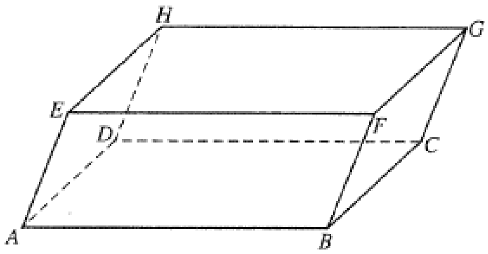
6 Find the area of the parallelogram $ABCD$ where A is the point with coordinates $(1, 2, -3)$, B is the point with coordinates $(-1, 3, -4)$ and D is the point with coordinates $(1, 5, -2)$.



7 Find the area of the parallelogram $ABCD$ in which the vertices A, B and D have coordinates $(-1, 2, 1)$, $(3, 1, 2)$ and $(5, 1, -6)$ respectively.

8 Find the area of the triangle with vertices $A(3, -1, 2)$, $B(1, -1, 3)$ and $C(4, -3, 1)$.

9



Find the volume of the parallelepiped $ABCDEFGH$ where the vertices A, B, D and E have coordinates $(0, 0, 0)$, $(5, -2, 3)$, $(2, -3, 4)$ and $(3, -1, -2)$ respectively.

10 The points A, B, C, D have position vectors

$$\mathbf{a} = (2\mathbf{i} + \mathbf{j}) \qquad \mathbf{b} = (3\mathbf{i} - \mathbf{j} + \mathbf{k})$$
$$\mathbf{c} = (-2\mathbf{j} - \mathbf{k}) \qquad \mathbf{d} = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k})$$

respectively.

(a) Find $\overrightarrow{AB} \times \overrightarrow{BC}$ and $\overrightarrow{BD} \times \overrightarrow{DC}$.

(b) Hence find

(i) the area of $\triangle ABC$

(ii) the volume of the tetrahedron $ABCD$. [L]

11 Relative to an origin O, the points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, where

$$\mathbf{p} = a(\mathbf{i} + \mathbf{j} + 2\mathbf{k}) \quad \text{and} \quad \mathbf{q} = a(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \text{ and } a > 0.$$

Find the area of triangle OPQ. [L]

12 Referred to an origin O, the points P and Q have position vectors $3\mathbf{i} - 3\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$ respectively. Find

(a) $\overrightarrow{OP}.\overrightarrow{OQ}$

(b) $\overrightarrow{OP} \times \overrightarrow{OQ}$

(c) the size, in degrees to 0.1°, of $\angle POQ$

(d) the area of $\triangle OPQ$.

13 Referred to O as origin, $\overrightarrow{OA} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\overrightarrow{OB} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OC} = -\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

(a) Show that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{OC}$.

(b) Find the area of $\triangle OAB$.

(c) Calculate the area of $\triangle ABC$.

14 The points $A(1, -1, -1)$, $B(-1, 1, -1)$, $C(-1, -1, 1)$ and $D(1, 1, 1)$ are given referred to a fixed origin O.

(a) Show that $ABCD$ is a regular tetrahedron.

(b) Find the volume of $ABCD$.

15 Find the volume of the tetrahedron with vertices at the points $(1, 3, -1)$, $(2, 2, 3)$, $(4, 2, -2)$ and $(3, 7, 4)$.

16 A tetrahedron has its vertices at the points $O(0, 0, 0)$, $A(-1, 1, 2)$, $B(1, 2, -1)$ and $C(0, 1, 3)$.

(a) Determine the area of the face ABC.

(b) Find a unit vector normal to the face ABC.

(c) Find the volume of the tetrahedron.

17 Find the volume of the tetrahedron with vertices $(0, 1, 0)$, $(0, 0, -4)$, $(2, -1, 3)$, $(2, -1, 2)$.

18 The tetrahedron $ABCD$ has vertices $A(1,-1,0)$, $B(0,2,-1)$, $C(0,2,1)$, $D(-1,3,0)$.

(a) Find the area of face BCD.

(b) Find the volume of the tetrahedron.

19 A tetrahedron $OABC$ has its vertices at the points $O(0,0,0)$, $A(1,2,-1)$, $B(-1,1,2)$ and $C(2,-1,1)$.

(a) Write down expressions for $\overrightarrow{AB}$ and $\overrightarrow{AC}$ in terms of **i**, **j** and **k** and find $\overrightarrow{AB} \times \overrightarrow{AC}$.

(b) Deduce the area of $\triangle ABC$.

(c) Find the volume of the tetrahedron $OABC$. [L]

20 The edges OP, OQ, OR of a tetrahedron $OPQR$ are the vectors **a**, **b** and **c** respectively, where

$$\mathbf{a} = 2\mathbf{i} + 4\mathbf{j}$$
$$\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$
$$\mathbf{c} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$$

(a) Evaluate $\mathbf{b} \times \mathbf{c}$ and deduce that OP is perpendicular to the plane OQR.

(b) Write down the length of OP and the area of $\triangle OQR$ and hence the volume of the tetrahedron.

(c) Verify your result by evaluating $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$. [L]

ANSWERS

Exercise 6B

1 $\frac{5\sqrt{2}}{2}$ **2** $\frac{13\sqrt{3}}{2}$ **3** $\frac{\sqrt{11}}{2}$ **4** $\sqrt{11}$

5 $\frac{5\sqrt{10}}{2}$ **6** $2\sqrt{14}$ **7** $6\sqrt{34}$ **8** $\frac{\sqrt{21}}{2}$

9 39

10 (a) $5\mathbf{i} - \mathbf{j} - 7\mathbf{k}$, $2\mathbf{i} - 8\mathbf{j} + \mathbf{k}$

(b) (i) $\frac{5\sqrt{3}}{2}$ (ii) $\frac{19}{6}$

11 $\frac{9\sqrt{3}}{2}$

12 (a) 24 (b) $6\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$

(c) $39.7°$ (d) $3\sqrt{11}$

13 (b) $\frac{1}{2}\sqrt{171}$ (c) $\frac{3}{2}\sqrt{133}$

14 (b) $\frac{8}{3}$ **15** 12

16 (a) $\frac{3}{2}\sqrt{3}$ (b) $\pm\frac{1}{\sqrt{27}}(\mathbf{i} - 5\mathbf{j} - \mathbf{k})$

(c) $\frac{4}{3}$

17 $\frac{1}{3}$

18 (a) $\sqrt{2}$ (b) $\frac{2}{3}$

19 (a) $-2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$; $7\mathbf{i} + 7\mathbf{j} + 7\mathbf{k}$

(b) $\frac{7}{2}\sqrt{3}$ (c) $\frac{7}{3}$

20 (a) $\mathbf{i} + 2\mathbf{j}$ (b) $2\sqrt{5}$, $\frac{1}{2}\sqrt{5}$; $\frac{5}{3}$

*P4 book page 228 Ex6C Q1ab,2ab,3ab,4ab,5ab,6a,7ac,8ac,9-12

1 Find, in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, an equation of the straight line passing through the point with position vector $\mathbf{a}$ and which is parallel to the vector $\mathbf{b}$ where:

(a) $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{b} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

(b) $\mathbf{a} = \mathbf{i} - 2\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$

(c) $\mathbf{a} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

(d) $\mathbf{a} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

2 Find, in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, an equation of the line passing through the points with coordinates

(a) (3, 2, 5), (9, 5, 17)

(b) (1, −2, −1), (2, −4, −4)

(c) (−4, −3, 11), (0, −3, 1)

(d) (3, −2, 3), (−1, 0, 1)

3 Find an equation, in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$, of the line given by the equation, where λ is a scalar:

(a) $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

(b) $\mathbf{r} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$

(c) $\mathbf{r} = \mathbf{i} - 4\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$

(d) $\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \lambda(-3\mathbf{i} + 2\mathbf{j})$

4 Find in the form $\mathbf{r}.\mathbf{n} = p$ an equation of the plane that passes through the point with position vector $\mathbf{a}$ and is perpendicular to the vector $\mathbf{n}$ where:

(a) $\mathbf{a} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{n} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

(b) $\mathbf{a} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}, \quad \mathbf{n} = -\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

(c) $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}, \quad \mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

(d) $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \quad \mathbf{n} = -3\mathbf{i} + \mathbf{k}$

(e) $\mathbf{a} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}, \quad \mathbf{n} = 6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

5 Find a cartesian equation for each of the planes in question 4.

6 Verify that the point with position vector **a** lies in the given plane (λ, μ scalars) where:

(a) $\mathbf{a} = 2\mathbf{j} + \mathbf{k}$, $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(2\mathbf{i} - \mathbf{j} + \mathbf{k})$

(b) $\mathbf{a} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{r} = \mathbf{i} + 3\mathbf{j} - \mathbf{k} + \lambda(-\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + \mu(-\mathbf{i} - \mathbf{j} - \mathbf{k})$

(c) $\mathbf{a} = 24\mathbf{i} + 25\mathbf{j} - 9\mathbf{k}$, $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(7\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}) + \mu(3\mathbf{i} - 4\mathbf{j} + \mathbf{k})$

7 Find (i) an equation of the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ (ii) a cartesian equation of the plane passing through the points:

(a) (1, −1, 1), (2, −4, 3), (0, 1, −3)

(b) (4, 7, −1), (1, 1, −4), (2, −2, 3)

(c) (8, 1, −1), (2, 6, −2), (3, −3, 0)

(d) (2, 0, −3), (1, 4, −1), (2, −1, 0)

8 Find a cartesian equation of the plane containing the points:

(a) (1, 1, 1), (2, 1, 0), (2, 2, −1)

(b) (2, 1, −1), (−2, −1, −5), (0, −4, 3)

(c) (1, 1, 2), (3, 4, 1), (−5, 1, −1)

(d) (4, 0, 0), (0, 3, 0), $\left(0, 0, -\frac{1}{2}\right)$

9 Find the coordinates of the point of intersection of the line l and the plane Π where

(a) l: $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + t(-\mathbf{i} + 2\mathbf{k})$, t scalar

Π: $\mathbf{r}.(2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 4$

(b) l: $\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k} + t(2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$, t scalar

Π: $\mathbf{r}.(6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) = 10$

10 Find an equation of the plane, in the form $\mathbf{r}.\mathbf{n} = p$, which contains the line l and the point with position vector **a** where

(a) l: $\mathbf{r} = t(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$, $\mathbf{a} = \mathbf{i} + 4\mathbf{k}$

(b) l: $\mathbf{r} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k} + t(-\mathbf{i} + \mathbf{j} + 4\mathbf{k})$, $\mathbf{a} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

(c) l: $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 3\mathbf{k})$, $\mathbf{a} = -3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

11 Find a cartesian equation of the plane which passes through the origin O and contains the line with equations

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

12 Referred to an origin O, the points A, B, C have coordinates $(3, 2, 0)$, $(1, 0, 1)$, $(2, 2, 2)$ respectively.

(a) Find a cartesian equation of the plane ABC.

(b) Show that $D(4, 4, 1)$ lies in the plane.

(c) Show that AB and DC are parallel.

(d) Find the coordinates of the point where the lines AC and BD meet.

ANSWERS

Exercise 6C

1 (a) $\mathbf{r} \times (-\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

(b) $\mathbf{r} \times (2\mathbf{i} + 3\mathbf{j}) = 6\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

(c) $\mathbf{r} \times (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 13\mathbf{i} + 7\mathbf{j} + 11\mathbf{k}$

(d) $\mathbf{r} \times (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 7\mathbf{i} + 11\mathbf{j} + \mathbf{k}$

2 (a) $\left[\mathbf{r} - \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}\right] \times \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = \mathbf{0}$

(b) $\left[\mathbf{r} - \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}\right] \times \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = \mathbf{0}$

(c) $\left[\mathbf{r} - \begin{pmatrix} -4 \\ -3 \\ 11 \end{pmatrix}\right] \times \begin{pmatrix} 4 \\ 0 \\ -10 \end{pmatrix} = \mathbf{0}$

(d) $\left[\mathbf{r} - \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}\right] \times \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = \mathbf{0}$

3 (a) $\left[\mathbf{r} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right] \times \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \mathbf{0}$

(b) $\left[\mathbf{r} - \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}\right] \times \begin{pmatrix} -1 \\ -3 \\ -4 \end{pmatrix} = \mathbf{0}$

(c) $\left[\mathbf{r} - \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}\right] \times \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \mathbf{0}$

(d) $\left[\mathbf{r} - \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}\right] \times \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} = \mathbf{0}$

4 (a) $\mathbf{r} . (2\mathbf{i} + 4\mathbf{j} - \mathbf{k}) = -4$

(b) $\mathbf{r} . (-\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = -22$

(c) $\mathbf{r} . (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = 11$

(d) $\mathbf{r} . (-3\mathbf{i} + \mathbf{k}) = -4$

(e) $\mathbf{r} . (6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 14$

5 (a) $2x + 4y - z = -4$

(b) $-x + 3y - 4z = -22$

(c) $2x + 3y - 4z = 11$

(d) $-3x + z = -4$

(e) $3x + 2y - z = 7$

7 (a) (i) $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$

(ii) $8x + 2y - z = 5$

(b) (i) $\mathbf{r} = \begin{pmatrix} 4 \\ 7 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -9 \\ 4 \end{pmatrix}$

(ii) $17x - 6y - 5z = 31$

(c) (i) $\mathbf{r} = \begin{pmatrix} 8 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix}$

(ii) $x + 11y + 49z = -30$

(d) (i) $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}$

(ii) $14x + 3y + z = 25$

8 (a) $x + y + z = 3$

(b) $7x - 6y - 4z = 12$

(c) $3x - 4y - 6z + 13 = 0$

(d) $3x + 4y - 24z = 12$

9 (a) $(0, -2, 3)$ (b) $(3, 1, 2)$

10 (a) $\mathbf{r} . (4\mathbf{i} - 3\mathbf{j} - \mathbf{k}) = 0$

(b) $\mathbf{r} . \begin{pmatrix} 4 \\ -16 \\ 5 \end{pmatrix} = -10$

(c) $\mathbf{r} . \begin{pmatrix} 9 \\ 3 \\ 5 \end{pmatrix} = -14$

11 $x - 2y + z = 0$

12 (a) $4x - 3y + 2x = 6$

(d) $\left(2\frac{1}{2}, 2, 1\right)$

*P4 book page 236 Ex6D Q1ac,2ac,3,4ab,5ab,6,7,8ab,11,13-17,19,20,21

1 Find the distance from the origin to the plane with equation:

(a) $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-3\mathbf{k})=5$

(b) $\mathbf{r}.(-3\mathbf{i}+\mathbf{j}+6\mathbf{k})=24$

(c) $\mathbf{r}.(5\mathbf{i}-10\mathbf{j}+4\mathbf{k})=17$

(d) $\mathbf{r}.(3\mathbf{i}+12\mathbf{j}-4\mathbf{k})=62$

(e) $x-y+2z=15$

2 Find the distance from the given point to the given plane:

(a) $(1,2,-3)$, $\mathbf{r}.(\mathbf{i}+2\mathbf{j}-\mathbf{k})=23$

(b) $(1,-3,2)$, $\mathbf{r}.(2\mathbf{i}-\mathbf{j}+3\mathbf{k})=46$

(c) $(4,1,-7)$, $2x+6y-3z=14$

(d) $(1,-3,5)$, $\mathbf{r}.(4\mathbf{i}-2\mathbf{j}-4\mathbf{k})=10$

(e) $(4,-8,-1)$, $4x+y-7z=42$

3 Find the position vector of the point where the line with equation $\mathbf{r}=\mathbf{i}+\mathbf{j}-2\mathbf{k}+\lambda(\mathbf{i}+5\mathbf{k})$ cuts the plane with equation $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-\mathbf{k})=2$.

4 Find, in degrees to 0.1°, the acute angle between the given line and plane:

(a) $\mathbf{r}=\mathbf{i}-2\mathbf{k}+\lambda(\mathbf{i}+\mathbf{j}-3\mathbf{k})$ and $\mathbf{r}.(2\mathbf{i}-\mathbf{j}+4\mathbf{k})=10$

(b) $\mathbf{r}=3\mathbf{i}+\mathbf{j}-\mathbf{k}+\lambda(4\mathbf{i}-7\mathbf{k})$ and $\mathbf{r}.(-\mathbf{i}+4\mathbf{j}-6\mathbf{k})=24$

(c) $\mathbf{r}=-6\mathbf{i}+2\mathbf{j}-4\mathbf{k}+\lambda(-2\mathbf{i}+3\mathbf{j}-6\mathbf{k})$ and $\mathbf{r}.(2\mathbf{i}-6\mathbf{j}+5\mathbf{k})=63$

5 Find, in degrees to 0.1°, the angle between the planes with equations:

(a) $\mathbf{r}.(\mathbf{i}+2\mathbf{j}-2\mathbf{k})=3$ and $\mathbf{r}.(2\mathbf{i}+2\mathbf{j}-\mathbf{k})=6$

(b) $\mathbf{r}.(\mathbf{i}-\mathbf{j}-\sqrt{2}\mathbf{k})=5$ and $\mathbf{r}.(7\mathbf{j}+\mathbf{k})=10$

(c) $x-2y-5z=7$ and $3x+7y-z=4$

6 Find the distance from the origin to the plane with equation $\mathbf{r}.(2\mathbf{i}-2\mathbf{j}+\mathbf{k})=6$.

7 Find the cosine of the acute angle between the planes with equations $2x+3y-4z=5$ and $6x-2y-3z=4$.

8 Find a vector equation of the line of intersection of the planes with equations:

(a) $\mathbf{r}.(2\mathbf{i}+\mathbf{j}-3\mathbf{k})=28$ and $\mathbf{r}.(4\mathbf{i}-7\mathbf{j}+\mathbf{k})=31$

(b) $x+6y+z=-10, \quad 3x+2y-z=-1$

(c) $\mathbf{r}.(\mathbf{i}+2\mathbf{j}+\mathbf{k})=-5$ and $\mathbf{r}.(2\mathbf{i}-3\mathbf{j}+\mathbf{k})=15$

9 Find, in degrees to 0.1°, the acute angle between the planes with equations $3x+4y+2z=7$ and $2x-3y+z=9$.

10 Find the distance of the origin from the plane with equation $\mathbf{r}.(3\mathbf{i}-4\mathbf{j}-12\mathbf{k})=26$.

11 (a) Find, in cartesian form, the equation of the plane Π which passes through the origin and contains the line with equations

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

(b) Find, in degrees to 0.1°, the acute angle between Π and the plane with equation

$$4x+y-z=3$$

12 A line has equation

$$\mathbf{r}=3\mathbf{i}-5\mathbf{j}+2\mathbf{k}+\lambda(2\mathbf{i}-4\mathbf{j}+\mathbf{k})$$

and a plane has equation

$$\mathbf{r}.(3\mathbf{i}-\mathbf{j}-5\mathbf{k})=1$$

Find the acute angle between the line and the plane.

13 The planes with equations

$$2x-y+3z+3=0 \quad \text{and} \quad x+10y=21$$

meet in a line L.

The planes with equations

$$2x-y=0 \quad \text{and} \quad 7x+z=6$$

meet in a line M.

Show that L and M meet at a point. Show further that L and M both lie in the plane with equation $x+3y+z=6$.

14 Referred to a fixed origin O, the lines l_1 and l_2 have equations

$$\mathbf{r} = 3\mathbf{i} + 6\mathbf{j} + \mathbf{k} + s(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

and

$$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

respectively, where s and t are scalar parameters.

(a) Show that l_1 and l_2 intersect and determine the position vector of their point of intersection.

(b) Show that the vector $-\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ is perpendicular to both l_1 and l_2.

(c) Find, in the form $\mathbf{r}.\mathbf{n} = p$, an equation of the plane containing l_1 and l_2. [L]

15 With respect to the origin O the points A, B, C have position vectors

$$5\mathbf{i} - \mathbf{j} - 3\mathbf{k}, \quad -4\mathbf{i} + 4\mathbf{j} - \mathbf{k}, \quad 5\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}$$

respectively. Find

(a) a vector equation for the line BC

(b) a vector equation for the plane OAB

(c) the cosine of the acute angle between the lines OA and OB.

Obtain, in the form $\mathbf{r}.\mathbf{n} = p$, a vector equation for Π, the plane which passes through A and is perpendicular to BC.

Find cartesian equations for

(d) the plane Π

(e) the line BC. [L]

16 Show that

$$\mathbf{r} = \mathbf{a} + s(\mathbf{b} - \mathbf{a}) + t(\mathbf{c} - \mathbf{a})$$

is an equation of the plane which passes through the non-collinear points whose position vectors are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, where $\mathbf{r}$ is the position vector of a general point on the plane and s and t are scalars. Find a cartesian equation of the plane containing the points $(1, 1, -1)$, $(2, 0, 1)$ and $(3, 2, 1)$, and show that the points $(2, 1, 2)$ and $(0, -2, -2)$ are equidistant from, and on opposite sides of, this plane. [L]

17 A plane passes through the three points A, B, C whose position vectors, referred to an origin O, are $(\mathbf{i} + 3\mathbf{j} + 3\mathbf{k})$, $(3\mathbf{i} + \mathbf{j} + 4\mathbf{k})$, $(2\mathbf{i} + 4\mathbf{j} + \mathbf{k})$ respectively. Find, in the form $(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$, a unit vector normal to this plane. Find also a cartesian equation of the plane, and the perpendicular distance from the origin to this plane. [L]

18 Show that the vector $\mathbf{i} + \mathbf{k}$ is perpendicular to the plane with vector equation

$$\mathbf{r} = \mathbf{i} + s\mathbf{j} + t(\mathbf{i} - \mathbf{k})$$

Find the perpendicular distance from the origin to this plane. Hence, or otherwise, obtain a cartesian equation of the plane. [L]

19 Three planes have equations

$$x - 6y - z = 5$$
$$3x + 2y + z = -1$$
$$5x + pz = q$$

Show that

(a) the planes have a common point of intersection unless $p = 1$

(b) when $p = 1$, $q = 2$, the planes intersect in pairs in three parallel lines

(c) when $p = 1$, $q = 1$, the planes have a common line of intersection.

Give equations for the line of intersection in (c). [L]

20 Show that the lines l_1, l_2, with vector equations

$$\mathbf{r} = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} + \lambda(-3\mathbf{i} + \mathbf{j} - \mathbf{k})$$
$$\mathbf{r} = 10\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} + \mu(4\mathbf{i} - \mathbf{j} + 2\mathbf{k})$$

respectively, intersect and find a vector equation of the plane Π containing l_1 and l_2.

Show that the point Q with position vector $(6\mathbf{i} + 7\mathbf{j} - 2\mathbf{k})$ lies on the line which is perpendicular to Π and which passes through the intersection of l_1 and l_2. Find a vector equation of the plane which passes through Q and is parallel to Π. [L]

21 With respect to a fixed origin O, the straight lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \mathbf{i} - \mathbf{j} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$
$$l_2 : \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \mu(-3\mathbf{i} + 4\mathbf{k})$$

where λ and μ are scalar parameters.

(a) Show that the lines intersect.

(b) Find the position vector of their point of intersection.

(c) Find the cosine of the acute angle contained between the lines.

(d) Find a vector equation of the plane containing the lines. [L]

22 The position vectors of the points A, B, C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where

$$\mathbf{a} = -2\mathbf{i} + \mathbf{j}, \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}, \quad \mathbf{c} = 5\mathbf{j} - 4\mathbf{k}$$

(a) Find $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$ and hence, or otherwise, find an equation of the plane ABC in the form $\mathbf{r}.\mathbf{n} = p$ and the area of the triangle ABC.

(b) Find a vector equation of the plane which passes through A and which is perpendicular to both the plane ABC and the plane with equation $(\mathbf{r} - \mathbf{a}).\mathbf{b} = 0$.

(c) Find the cartesian equations of the line BC. [L]

23 Planes Π_1 and Π_2 have equations given by

$$\Pi_1 : \mathbf{r}.(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$$
$$\Pi_2 : \mathbf{r}.(\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}) = 1$$

(a) Show that the point $A(2, -2, 3)$ lies in Π_2.

(b) Show that Π_1 is perpendicular to Π_2.

(c) Find, in vector form, an equation of the straight line through A which is perpendicular to Π_1.

(d) Determine the coordinates of the point where this line meets Π_1.

(e) Find the perpendicular distance of A from Π_1.

(f) Find a vector equation of the plane through A parallel to Π_1. [L]

24 Show that the line with equations

$$\frac{x-4}{1} = \frac{y-5}{2} = \frac{z-6}{3}$$

and the line with equations

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}$$

intersect.

Find an equation for the plane in which they lie and the coordinates of their point of intersection. [L]

ANSWERS

Exercise 6D

1 (a) $\frac{5}{14}\sqrt{14}$ (b) $\frac{12}{23}\sqrt{46}$ (c) $\frac{17}{141}\sqrt{141}$
(d) $\frac{62}{13}$ (e) $\frac{5}{2}\sqrt{6}$

2 (a) $\frac{5}{2}\sqrt{6}$ (b) $\frac{5}{2}\sqrt{14}$ (c) 3
(d) $3\frac{1}{3}$ (e) $\frac{9}{22}\sqrt{66}$

3 $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$

4 (a) 46.4° (b) 40.3° (c) 67.1°

5 (a) 27.3° (b) 53.5° (c) 81.8°

6 2

7 $\dfrac{18\sqrt{29}}{203}$

8 (a) $\mathbf{r}=\frac{121}{14}\mathbf{i}-\frac{25}{7}\mathbf{k}+\lambda\left(\frac{10}{7}\mathbf{i}+\mathbf{j}+\frac{9}{7}\mathbf{k}\right)$
(b) $\mathbf{r}=-\frac{11}{8}\mathbf{j}-\frac{7}{4}\mathbf{k}+\lambda\left(\mathbf{i}-\frac{1}{2}\mathbf{j}+2\mathbf{k}\right)$
(c) $\mathbf{r}=20\mathbf{i}-25\mathbf{k}+\lambda(5\mathbf{i}+\mathbf{j}-7\mathbf{k})$

9 78.5°

10 2

11 (a) $x-2y+z=0$ (b) 84.5°

12 10.6°

14 (a) $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$
(c) $\mathbf{r}\cdot\begin{pmatrix}-1\\3\\7\end{pmatrix}=22$

15 (a) $\mathbf{r}=\begin{pmatrix}-4\\4\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\-2\\4\end{pmatrix}$
(b) $\mathbf{r}=s\begin{pmatrix}5\\-1\\-3\end{pmatrix}+t\begin{pmatrix}-4\\4\\-1\end{pmatrix}$
(c) $\sqrt{\frac{21}{55}}$; $\mathbf{r}\cdot\begin{pmatrix}3\\-2\\4\end{pmatrix}=5$
(d) $3x-2y+4z=5$
(e) $\dfrac{x+4}{3}=\dfrac{y-4}{-2}=\dfrac{z+1}{4}$

16 $4x-2y-3z=5$

17 $\frac{1}{5\sqrt{2}}(3\mathbf{i}+5\mathbf{j}+4\mathbf{k})$; $3x+5y+4z=30$; $3\sqrt{2}$

18 $\frac{1}{2}\sqrt{2}$; $x+z=1$

19 (c) $\dfrac{x}{1}=\dfrac{y+1}{1}=\dfrac{z-1}{-5}$

20 $\mathbf{r}=\begin{pmatrix}2\\-1\\2\end{pmatrix}+s\begin{pmatrix}-3\\1\\-1\end{pmatrix}+t\begin{pmatrix}4\\-1\\2\end{pmatrix}$
$\mathbf{r}=\begin{pmatrix}6\\7\\-2\end{pmatrix}+p\begin{pmatrix}-3\\1\\-1\end{pmatrix}+q\begin{pmatrix}4\\-1\\2\end{pmatrix}$

21 (b) $7\mathbf{i}+2\mathbf{j}-6\mathbf{k}$ (c) $\frac{14}{15}$
(d) $\mathbf{r}=\begin{pmatrix}7\\2\\-6\end{pmatrix}+s\begin{pmatrix}2\\1\\-2\end{pmatrix}+t\begin{pmatrix}-3\\0\\4\end{pmatrix}$

22 (a) $4\mathbf{i}+8\mathbf{j}+10\mathbf{k}$; $\mathbf{r}\cdot\begin{pmatrix}4\\8\\10\end{pmatrix}=0$; $3\sqrt{5}$
(b) $\mathbf{r}=\begin{pmatrix}-2\\1\\0\end{pmatrix}+s\begin{pmatrix}4\\8\\10\end{pmatrix}+t\begin{pmatrix}1\\2\\-2\end{pmatrix}$
(c) $\dfrac{x-1}{-1}=\dfrac{y-2}{3}=\dfrac{z+2}{-2}$ $(=\lambda)$

23 (c) $\mathbf{r}=\begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$
(d) $\left(-1,\ -\frac{1}{2},\ 1\frac{1}{2}\right)$
(e) $\frac{3}{2}\sqrt{6}$
(f) $\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=9$

24 $\mathbf{r}=\begin{pmatrix}5\\7\\9\end{pmatrix}+s\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}4\\5\\6\end{pmatrix}$; (5, 7, 9)