* Complex Numbers Video 1

Complex Numbers

The quadratic equation $ax^2 + bx + c = 0$ has solutions given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

* $\sqrt{-1} = i$

If the expression under the square root is negative, there are no real solutions.

You can find solutions to the equation in all cases by extending the number system to include $\sqrt{-1}$. Since there is no real number that squares to produce -1, the number $\sqrt{-1}$ is called an **imaginary number**, and is represented using the letter **i**. Sums of real and imaginary numbers, for example $3 + 2i$, are known as **complex numbers**.

- **$i = \sqrt{-1}$**
- **An imaginary number is a number of the form bi, where $b \in \mathbb{R}$.**
- **A complex number is written in the form $a + bi$, where $a, b \in \mathbb{R}$.**

Notation The set of all complex numbers is written as $\mathbb{C}$.
For the complex number $z = a + bi$:
- $\text{Re}(z) = a$ is the real part
- $\text{Im}(z) = b$ is the imaginary part

In a complex number, the real part and the imaginary part cannot be combined to form a single term.

- **Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.**
- **You can multiply a real number by a complex number by multiplying out the brackets in the usual way.**

Example

Simplify each of the following, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

a $(2 + 5i) + (7 + 3i)$ **b** $(2 - 5i) - (5 - 11i)$ **c** $2(5 - 8i)$ **d** $\dfrac{10 + 6i}{2}$

Solution

(a) $9 + 8i$ (b) $2 - 5i - 5 + 11i$
$= -3 + 6i$ (c) $2(5 - 8i)$
$= 10 - 16i$ (d) $\dfrac{10 + 6i}{2}$
$= 5 + 3i$

You can use complex numbers to find solutions to any quadratic equation with real coefficients.

- **If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which are real.**

Example

Solve the equation $z^2 + 9 = 0$.

Solution

$z^2 = -9$

$z = \pm\sqrt{-9}$

$z = \pm\sqrt{9}\sqrt{-1}$

$= \pm 3i$

$* \sqrt{-1} = i$

Example

Solve the equation $z^2 + 6z + 25 = 0$.

Solution

$$z = \frac{-6 \pm \sqrt{(6)^2 - 4(1)(25)}}{2}$$

$$z = \frac{-6 \pm \sqrt{-64}}{2}$$

$$z = \frac{-6 \pm \sqrt{64}\sqrt{-1}}{2}$$

$$z = \frac{-6 \pm 8i}{2}$$

$$z = -3 \pm 4i$$

Multiplying complex numbers

You can multiply complex numbers using the same technique that you use for multiplying brackets in algebra. You can use the fact that $i = \sqrt{-1}$ to simplify powers of i.

- $i^2 = -1$

Example

Express each of the following in the form $a + bi$, where a and b are real numbers.

a $(2 + 3i)(4 + 5i)$ **b** $(7 - 4i)^2$

Solution

(a) $(2+3i)(4+5i)$

$= 8 + 10i + 12i + 15i^2$

$= 8 + 22i + 15(-1)$

$= 8 + 22i - 15$

$= -7 + 22i$

(b) $(7-4i)(7-4i)$

$= 49 - 28i - 28i + 16i^2$

$= 49 - 56i - 16$

$= 33 - 56i$

Complex conjugation

- **For any complex number $z = a + bi$, the complex conjugate of the number is defined as $z^* = a - bi$.**

Notation Together z and z^* are called a **complex conjugate pair**.

Example

Given that $z = 2 - 7i$,

a write down z^*

b find the value of $z + z^*$

c find the value of zz^*

Solution

(a) $z = 2 - 7i$

$z^* = 2 + 7i$

(b) $z + z^* = 2 - 7i + 2 + 7i$

$= 4$

(c) $zz^* = (2 - 7i)(2 + 7i)$

$= 4 - 49i^2$

$= 4 - 49(-1)$

$= 4 + 49$

$= 53$

Note The modulus of a complex number is denoted by $|z|$ $where$ $|z| = \sqrt{a^2 + b^2}$ i.e. $zz^* = |z|^2$.

Dividing Complex Numbers

you multiply both the numerator and the denominator by the complex conjugate of the denominator and then simplify the result.

Links The method used to divide complex numbers is similar to the method used to rationalise a denominator when simplifying surds.

Example

Write $\frac{5 + 4i}{2 - 3i}$ in the form $a + bi$.

Solution

$\frac{5+4i}{2-3i} \times \frac{2+3i}{2+3i}$

$= \frac{10 + 15i + 8i - 12}{4 - 9(-1)}$

$= \frac{-2 + 23i}{13}$

$= -\frac{2}{13} + \frac{23}{13}i$

Note two complex numbers are equal if and only if their real and imaginary parts are separately equal.

$$\text{e.g. If } z_1 = z_2$$

$$\therefore x_1 + iy_1 = x_2 + iy_2$$

$$\therefore (x_1 - x_2)^2 = [i(y_2 - y_1)]^2$$

$$\therefore (x_1 - x_2)^2 = -1(y_2 - y_1)^2$$

$$\therefore (x_1 - x_2)^2 + (y_2 - y_1)^2 = 0$$

$$\therefore (x_1 - x_2)^2 = 0 \text{ and } (y_2 - y_1)^2 = 0 \text{ (since the square of real numbers must be positive)}$$

$$\therefore x_1 = x_2 \text{ and } y_1 = y_2$$

Note If $x + iy = 0$ then $x = 0$ and $y = 0$

***P3 book page 166 EX7A Q1ace,2ace,3ace,4,5ace,7ace,8ace,9ace,11,12**

*Complex Numbers Video 2

Finding the square root of a complex number

Example Find $\sqrt{3+4i}$

Solution

let $\sqrt{3+4i} = a+ib$

square both sides $3+4i = (a+ib)^2$

$3+4i = (a+ib)(a+ib)$

$3+4i = a^2 + 2iab - b^2$

$3+4i = (a^2-b^2) + i(2ab)$

equate reals $3 = a^2 - b^2$ ... ①

equate imags $4 = 2ab$

$2 = ab$ ... ②

$b = \frac{2}{a}$

$\therefore$ ① $3 = a^2 - \left(\frac{2}{a}\right)^2$

$3 = a^2 - \frac{4}{a^2}$

$\times a^2$

$3a^2 = a^4 - 4$

$0 = a^4 - 3a^2 - 4$

$0 = (a^2-4)(a^2+1) = 0$

$a^2 = 4$ or $a^2 = -1$

$\therefore a = \pm 2$ *a is real so $a \neq \sqrt{-1}$

$\therefore b = \frac{2}{a}$

$b = \pm 1$

$\therefore \sqrt{3+4i} = 2+i$ or $-2-i$

***P3 book page 179 EX7C Q1ace,2ace,3,4,8**

* Complex Numbers Video 3

Roots of quadratic equations

- **For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are non-real complex numbers, then they occur as a conjugate pair.**

Another way of stating this is that for a real-valued quadratic function $f(z)$, if z_1 is a root of $f(z) = 0$ then $z_1{}^*$ is also a root. You can use this fact to find one root if you know the other, or to find the original equation.

- **If the roots of a quadratic equation are α and β, then you can write the equation as $(z - \alpha)(z - \beta) = 0$**

  or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$

Notation Roots of complex-valued polynomials are often written using Greek letters such as α (alpha), β (beta) and γ (gamma).

Example

Given that $\alpha = 7 + 2i$ is one of the roots of a quadratic equation with real coefficients,

a state the value of the other root, β

b find the quadratic equation

c find the values of $\alpha + \beta$ and $\alpha\beta$ and interpret the results.

Solution

(a) $\alpha = 7+2i$

$\beta = 7-2i$ as roots occur in conjugate pairs

(b) $(z-\alpha)(z-\beta)=0$

$(z-(7+2i))(z-(7-2i))=0$

$z^2-(7-2i)z-(7+2i)z+(7+2i)(7-2i)=0$

$z^2-7z+2iz-7z-2iz+49+4=0$

$z^2-14z+53=0$

(c) $\alpha+\beta = 7+2i+7-2i$

$=14$

$\alpha\beta = (7+2i)(7-2i)$

$=49-4(-1)$

$=53$

Note

If $z=a+ib$

$zz^*=a^2+b^2$

Solving cubic and quartic equations

You can generalise the rule for the roots of quadratic equations to any polynomial with real coefficients.

- **If $f(z)$ is a polynomial with real coefficients, and z_1 is a root of $f(z) = 0$, then $z_1{}^*$ is also a root of $f(z) = 0$.**

Note If z_1 is real, then $z_1{}^* = z_1$.

You can use this property to find roots of cubic and quartic equations with real coefficients.

- **An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.**
- **For a cubic equation with real coefficients, either:**
  - **all three roots are real, or**
  - **one root is real and the other two roots form a complex conjugate pair.**

Watch out A real-valued cubic equation might have two, or three, repeated real roots.

Example

Given that -1 is a root of the equation $z^3 - z^2 + 3z + k = 0$,

a find the value of k **b** find the other two roots of the equation.

Solution

(a) If $z=-1$ is a root

then $(-1)^3-(-1)^2+3(-1)+k=0$

$-1-1-3+k=0$

$k=5$

(b) $z^3-z^2+3z+5=0$

If $z=-1$ is a root

then $(z+1)$ is a factor.

$$\begin{array}{r|l} & z^2-2z+5 \\ z+1 & z^3-z^2+3z+5 \\ & z^3+z^2 \\ & -2z^2+3z \\ & -2z^2-2z \\ & 5z+5 \\ & 5z+5 \\ & 0 \end{array}$$

$\therefore z^3-z^2+3z+5=(z+1)(z^2-2z+5)$

solving z^2-2z+5 $\quad z=\dfrac{2\pm\sqrt{(-2)^2-4(1)(5)}}{2}$

$z=\dfrac{2\pm\sqrt{-16}}{2}=\dfrac{2\pm 4i}{2}$

$z=1\pm 2i$

$\rightarrow$ ans: roots are $z=-1,\ 1+2i,\ 1-2i$

- **An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.**
- **For a quartic equation with real coefficients, either:**
  - **all four roots are real, or**
  - **two roots are real and the other two roots form a complex conjugate pair, or**
  - **two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.**

Watch out A real-valued quartic equation might have repeated real roots or repeated complex roots.

Example

Given that $3 + i$ is a root of the quartic equation $2z^4 - 3z^3 - 39z^2 + 120z - 50 = 0$, solve the equation completely.

Solution ***use different method. Divide through by quadratic to reduce quartic to quadratic then solve that quadratic***

If $3+i$ is a root

then $3-i$ is also a root.

$(z-(3+i))(z-(3-i))$ is a factor

of $2z^4-3z^3-39z^2+120z-50=0$

$$(z-(3+i))(z-(3-i)) = z^2 - z(3-i) - z(3+i) + (9+1)$$
$$= z^2 - 6z + 10$$

$$\begin{array}{r l}
 & 2z^2 + 9z - 5 \\
z^2-6z+10 \,\big)\! & 2z^4 - 3z^3 - 39z^2 + 120z - 50 \\
 & 2z^4 - 12z^3 + 20z^2 \\
\hline
 & 9z^3 - 59z^2 + 120z \\
 & 9z^3 - 54z^2 + 90z \\
\hline
 & -5z^2 + 30z - 50 \\
 & -5z^2 + 30z - 50 \\
\hline
 & 0
\end{array}$$

$\therefore$ quartic becomes $(z^2-6z+10)(2z^2+9z-5)=0$

$S=9 \quad P=-10$

$$(z^2-6z+10)(2z^2+10z-z-5)=0$$
$$(z^2-6z+10)(2z(z+5)-1(z+5))=0$$
$$(z^2-6z+10)(2z-1)(z+5)=0$$

roots are: $3+i,\ 3-i,\ -5,\ \frac{1}{2}$

Example

Show that $z^2 + 4$ is a factor of $z^4 - 2z^3 + 21z^2 - 8z + 68$.

Hence solve the equation $z^4 - 2z^3 + 21z^2 - 8z + 68 = 0$.

Solution

$$
\begin{array}{r|l}
 & z^2 - 2z + 17 \\
\hline
z^2+4 & z^4 - 2z^3 + 21z^2 - 8z + 68 \\
 & z^4 \qquad\quad + 4z^2 \\
\hline
 & -2z^3 + 17z^2 - 8z \\
 & -2z^3 \qquad\qquad - 8z \\
\hline
 & 17z^2 \qquad\qquad + 68 \\
 & 17z^2 \qquad\qquad + 68 \\
\hline
 & 0
\end{array}
$$

So z^4+4 is a factor

rewrite polynomial as $(z^2+4)(z^2-2z+17)=0$

$z^2+4=0$

$z^2=-4$

$z=\pm\sqrt{-4}$

$z=\pm\sqrt{4}\sqrt{-1}$

$z=\pm 2i$

$z^2-2z+17=0$

$z=\dfrac{2\pm\sqrt{(-2)^2-4(1)(17)}}{2}$

$z=\dfrac{2\pm\sqrt{-64}}{2}$

$z=\dfrac{2\pm 8i}{2}$

$z=1\pm 4i$

ans: Roots are $-2i$, $2i$, $1-4i$, $1+4i$

*Edexcel Book E1F and Mixed Exercise 1 as extra

Summary of key points

1 $\mathbf{i} = \sqrt{-1}$ and $\mathbf{i}^2 = -1$

2 An **imaginary number** is a number of the form bi, where $b \in \mathbb{R}$.

3 A **complex number** is written in the form $a + b$i, where $a, b \in \mathbb{R}$.

4 Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.

5 You can multiply a real number by a complex number by multiplying out the brackets in the usual way.

6 If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which is real.

7 For any complex number $z = a + b$i, the **complex conjugate** of the number is defined as $z^* = a - b$i.

8 For real numbers a, b and c, if the roots of the quadratic equation $az^2 + bz + c = 0$ are non-real complex numbers, then they occur as a conjugate pair.

9 If the roots of a quadratic equation are α and β, then you can write the equation as $(z - \alpha)(z - \beta) = 0$ or $z^2 - (\alpha + \beta)z + \alpha\beta = 0$.

10 If f(z) is a polynomial with real coefficients, and z_1 is a root of f(z) = 0, then $z_1{}^*$ is also a root of f(z) = 0.

11 An equation of the form $az^3 + bz^2 + cz + d = 0$ is called a cubic equation, and has three roots.

For a cubic equation with real coefficients, either:

- all three roots are real, or
- one root is real and the other two roots form a complex conjugate pair.

12 An equation of the form $az^4 + bz^3 + cz^2 + dz + e = 0$ is called a quartic equation, and has four roots.

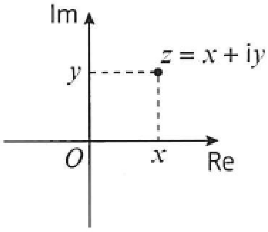
For a quartic equation with real coefficients, either:

- all four roots are real, or
- two roots are real and the other two roots form a complex conjugate pair, or
- two roots form a complex conjugate pair and the other two roots also form a complex conjugate pair.

*Complex Numbers Video 4

The Argand Diagram

- **You can represent complex numbers on an Argand diagram. The x-axis on an Argand diagram is called the real axis and the y-axis is called the imaginary axis. The complex number $z = x + iy$ is represented on the diagram by the point $P(x, y)$, where x and y are Cartesian coordinates.**



Complex numbers can also be represented as vectors on the Argand diagram.

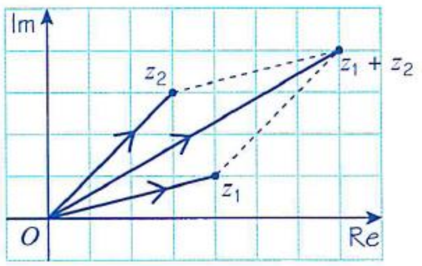
- **The complex number $z = x + iy$ can be represented as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on an Argand diagram.**

You can add or subtract complex numbers on an Argand diagram by adding or subtracting their corresponding vectors.

Example

$z_1 = 4 + i$ and $z_2 = 3 + 3i$. Show z_1, z_2 and $z_1 + z_2$ on an Argand diagram.

$z_1 + z_2 = (4 + 3) + (1 + 3)i = 7 + 4i$

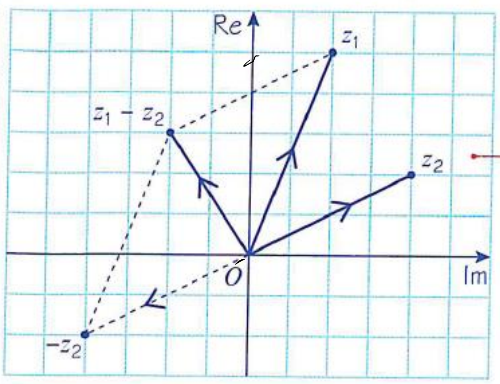


The vector representing $z_1 + z_2$ is the diagonal of the parallelogram with vertices at O, z_1 and z_2. You can use vector addition to find $z_1 + z_2$:
$\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

Example

$z_1 = 2 + 5i$ and $z_2 = 4 + 2i$. Show z_1, z_2 and $z_1 - z_2$ on an Argand diagram.

$z_1 - z_2 = (2 - 4) + (5 - 2)i = -2 + 3i$



The vector corresponding to z_2 is $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$, so the vector corresponding to $-z_2$ is $\begin{pmatrix} -4 \\ -2 \end{pmatrix}$.

The vector representing $z_1 - z_2$ is the diagonal of the parallelogram with vertices at O, z_1 and $-z_2$.

Modulus and argument

The **modulus** or absolute value of a complex number is the magnitude of its corresponding vector.

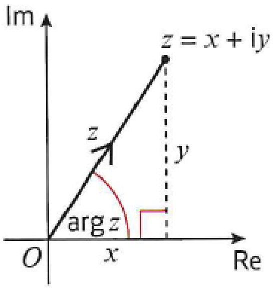
- **The modulus of a complex number, $|z|$, is the distance from the origin to that number on an Argand diagram. For a complex number $z = x + iy$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$.**

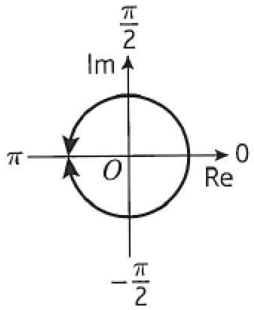
Notation The modulus of the complex number z is written as r, $|z|$ or $|x + iy|$.

The **argument** of a complex number is the angle its corresponding vector makes with the positive real axis.

- **The argument of a complex number, arg z, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram, measured in an anticlockwise direction. For a complex number $z = x + iy$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$**

Notation The argument of the complex number z is written as arg z. It is usually given in radians, where
- 2π radians = 360°
- π radians = 180°





The argument θ of any complex number is usually given in the range $-\pi < \theta \leqslant \pi$. This is sometimes referred to as the **principal argument**.

<u>Example</u>

$z = 2 + 7i$, find:

a the modulus of z **b** the argument of z, giving your answer in radians to 2 decimal places.

<u>Solution</u>

(a) $z = 2 + 7i$

$|z| = \sqrt{(2)^2 + (7)^2}$

$= \sqrt{4 + 49}$

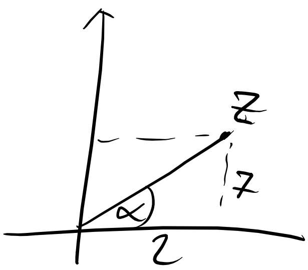
$= \sqrt{53}$

(b) $\arg z = \tan^{-1}\left(\frac{7}{2}\right)$

$= \tan^{-1}\left(\frac{7}{2}\right)$

$= 1.2924\ldots$

$= 1.29^c$



$\tan\alpha = \frac{7}{2}$

$\alpha = \tan^{-1}\left(\frac{7}{2}\right)$

Example

$z = -4 - \mathrm{i}$, find:

a the modulus of z

b the argument of z, giving your answer in radians to 2 decimal places.

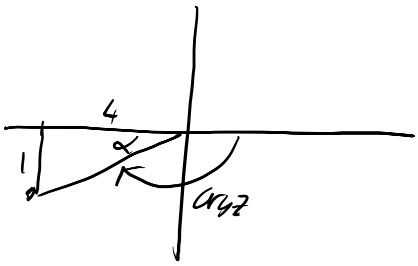
Solution

(a) $z = -4 - i$

$|z| = \sqrt{(-4)^2 + (-1)^2}$

$= \sqrt{17}$

(b)



$\tan\alpha = \frac{1}{4}$

$\alpha = 0.2449$

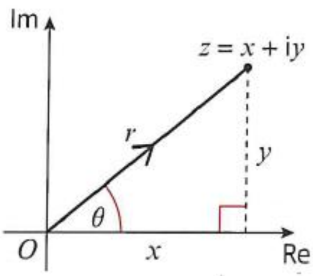
$\arg z = -(\pi - 0.2449)$

$= -2.90^c$ (2dp)

Modulus–argument form of complex numbers

You can write any complex number in terms of its modulus and argument.

- **For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus–argument form of z is $z = r(\cos\theta + \mathrm{i}\sin\theta)$**



From the right-angled triangle, $x = r\cos\theta$ and $y = r\sin\theta$.

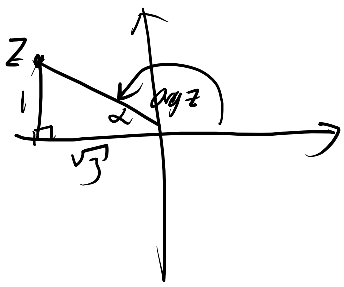
$z = x + \mathrm{i}y = r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta)$

This formula works for a complex number in any quadrant of the Argand diagram. The argument, θ, is usually given in the range $-\pi < \theta \leqslant \pi$, although the formula works for any value of θ measured anticlockwise from the positive real axis.

Example

Express $z = -\sqrt{3} + \mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $-\pi < \theta \leqslant \pi$.

Solution



$r = |z| = \sqrt{(-\sqrt{3})^2 + (1)^2}$

$= \sqrt{3+1}$

$= 2$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\alpha = \frac{\pi}{6}^c$

$\therefore \arg z = \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

ans: $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

*Edexcel Book Ex2A Q2-6

*Edexcel Book Ex2C Q1-3(alt parts),Q4-6

*Complex video 5

You can use the following rules to multiply complex numbers quickly when they are given in modulus–argument form.

- **For any two complex numbers z_1 and z_2,**
  - $|z_1z_2| = |z_1||z_2|$
  - **$\arg(z_1 z_2) = \arg z_1 + \arg z_2$**

Note You multiply the moduli and add the arguments.

To prove these results, consider z_1 and z_2 in modulus–argument form:

$$z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$$

Multiplying these numbers together, you get

$$\begin{aligned}
z_1z_2 &= r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \times r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)\\
&= r_1r_2(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 + \mathrm{i}\sin\theta_2)\\
&= r_1r_2(\cos\theta_1\cos\theta_2 + \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 + \mathrm{i}^2\sin\theta_1\sin\theta_2)\\
&= r_1r_2(\cos\theta_1\cos\theta_2 + \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)\\
&= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2))\\
&= r_1r_2(\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2))
\end{aligned}$$

Links The last step of this working makes use of the trigonometric addition formulae:
$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$
$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$

This complex number is in modulus–argument form, with modulus r_1r_2 and argument $\theta_1 + \theta_2$, as required.

You can derive similar results for dividing two complex numbers given in modulus–argument form.

- **For any two complex numbers z_1 and z_2,**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - **$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$**

Note You divide the moduli and subtract the arguments.

To prove these results, again consider z_1 and z_2 in modulus–argument form:

$$z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$$

Dividing z_1 by z_2 you get

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)}\\
&= \frac{r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)}{r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)} \times \frac{(\cos\theta_2 - \mathrm{i}\sin\theta_2)}{(\cos\theta_2 - \mathrm{i}\sin\theta_2)}\\
&= \frac{r_1(\cos\theta_1\cos\theta_2 - \mathrm{i}\cos\theta_1\sin\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 - \mathrm{i}^2\sin\theta_1\sin\theta_2)}{r_2(\cos\theta_2\cos\theta_2 - \mathrm{i}\cos\theta_2\sin\theta_2 + \mathrm{i}\sin\theta_2\cos\theta_2 - \mathrm{i}^2\sin\theta_2\sin\theta_2)}\\
&= \frac{r_1((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \mathrm{i}(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2))}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}\\
&= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \mathrm{i}\sin(\theta_1 - \theta_2))
\end{aligned}$$

Links The last step of this working makes use of the trigonometric addition formulae together with the identity $\sin^2\theta + \cos^2\theta \equiv 1$

This complex number is in modulus–argument form, with modulus $\frac{r_1}{r_2}$ and argument $\theta_1 - \theta_2$, as required.

Example

$z_1 = 3\left(\cos\frac{5\pi}{12} + \mathrm{i}\sin\frac{5\pi}{12}\right)$ and $4\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)$

a Find: **i** $|z_1 z_2|$ **ii** $\arg(z_1 z_2)$

b Hence write $z_1 z_2$ in the form: **i** $r(\cos\theta + \mathrm{i}\sin\theta)$ **ii** $x + \mathrm{i}y$

Solution

(a) (i) $|z_1 z_2| = |z_1||z_2|$

$= 3 \times 4$

$= 12$

(ii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$= \frac{5\pi}{12} + \frac{\pi}{12}$

$= 6\pi/12 = \pi/2$

(b) (i) $z_1 z_2 = \underline{12(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})}$

(ii) $z_1 z_2 = 12\cos\frac{\pi}{2} + i\, 12\sin\frac{\pi}{2}$

$= 12(0) + i\,12(1)$

$= 12i$

Example

Express $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + \mathrm{i}\sin\frac{5\pi}{6}\right)}$ in the form $x + \mathrm{i}y$.

Solution

$= \frac{\sqrt{2}}{2}\left(\cos\left(\pi/12 - 5\pi/6\right) + i\sin\left(\pi/12 - \frac{5\pi}{6}\right)\right)$

$= \frac{\sqrt{2}}{2}\left(\cos\left(-3\pi/4\right) + i\sin\left(-3\pi/4\right)\right)$

$= \sqrt{2}/2\left(-1/\sqrt{2} \quad i(-1/\sqrt{2})\right)$

$= -1/2 - 1/2\, i$

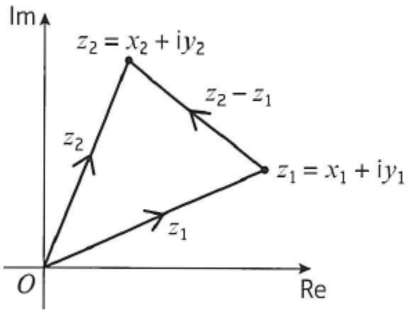
*Edexcel Book Ex2D Q1a,2,3ace,4ac

*Complex Numbers Video 6

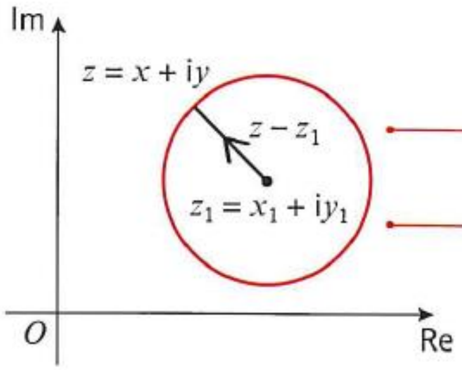
Loci in the Argand diagram

Complex numbers can be used to represent a locus of points on an Argand diagram.

- **For two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, $|z_2 - z_1|$ represents the distance between the points z_1 and z_2 on an Argand diagram.**



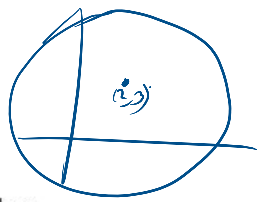
Using the above result, you can replace z_2 with the general point z. The locus of points described by $|z - z_1| = r$ is a circle with centre (x_1, y_1) and radius r.



Locus of points.

Every point z, on the circumference of the circle, is a distance of r from the centre of the circle.

eg. $|z-(2+3i)|=5$



- **Given $z_1 = x_1 + \mathrm{i}y_1$, the locus of point z on an Argand diagram such that $|z - z_1| = r$, or $|z - (x_1 + \mathrm{i}y_1)| = r$, is a circle with centre (x_1, y_1) and radius r.**

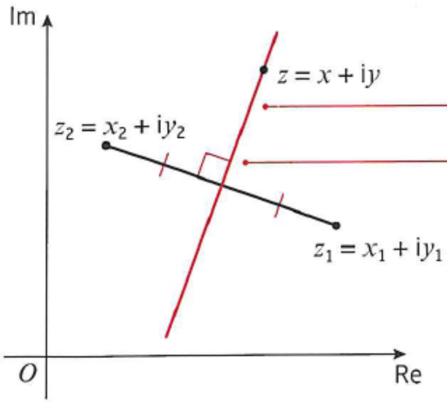
You can derive a Cartesian form of the equation of a circle from this form by squaring both sides:

$$|z - z_1| = r$$
$$|(x - x_1) + \mathrm{i}(y - y_1)| = r$$
$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

Since $|p + q\mathrm{i}| = \sqrt{p^2 + q^2}$

Links The Cartesian equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$

The locus of points that are an equal distance from two different points z_1 and z_2 is the perpendicular bisector of the line segment joining the two points.



Locus of points.

Every point z on the line is an equal distance from points z_1 and z_2.

- **Given $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, the locus of points z on an Argand diagram such that $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line segment joining z_1 and z_2.**

Example

Given that z satisfies $|z - 4| = 5$,

a sketch the locus of z on an Argand diagram.

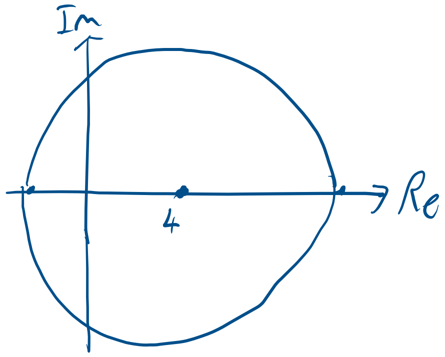
b Find the values of z that satisfy:

i both $|z - 4| = 5$ and $\text{Im}(z) = 0$ **ii** both $|z - 4| = 5$ and $\text{Re}(z) = 0$

Solution

(a) $|z-4|=5$

is a circle centre $(4,0)$ and radius $=5$



(b)(i) $\text{Im}(z)=0$ represents the real axis.

Pts where circle cuts real axis.

i.e. $(-1,0)$ and $(9,0)$

So values of z are $z=-1$ and $z=9$.

(ii) $|z-4|=5$ and $\text{Re}(z)=0$

$\text{Re}(z)=0$ is the Im. axis.

Put $x=0$ in $(x-4)^2+y^2=5^2$

i.e. $(-4)^2+y^2=25$

$16+y^2=25$

$y^2=9$

$y=\pm 3$

so pts circle crosses Im axis are $(0,-3)$ and $(0,3)$

ans: $z=-3i$ and $z=3i$

Example

A complex number z is represented by the point P in the Argand diagram.

Given that $|z - 5 - 3i| = 3$,

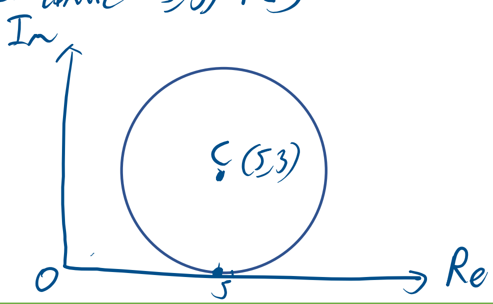
a sketch the locus of P **b** find the Cartesian equation of this locus

c find the maximum value of $\arg z$ in the interval $(-\pi, \pi)$.

Solution

(a) $|z-(5+3i)|=3$

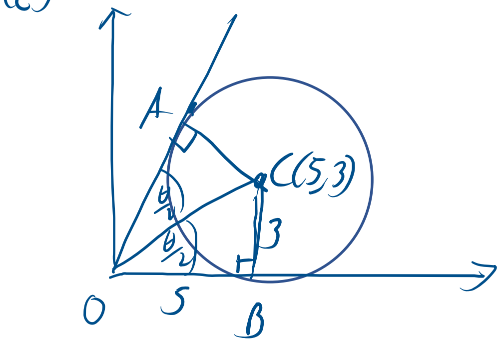
circle centre $(5,3)$ $r=3$



(b) $(x-5)^2+(y-3)^2=3^2$

$(x-5)^2+(y-3)^2=9$

(c)



Using Δ OBC

$\tan\left(\frac{\theta}{2}\right) = \frac{3}{5}$

$\frac{\theta}{2} = 0.5404195$

$\therefore \theta = 1.080839$

$\therefore$ max $\arg z = 1.08^c$ (2 d.p.)

Example

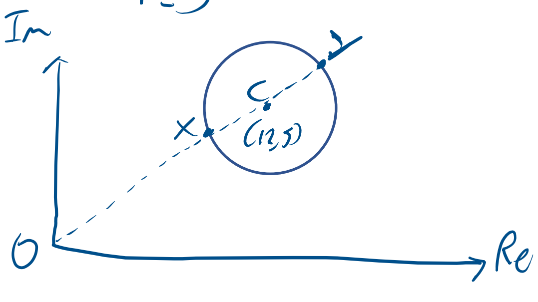
Given that the complex number $z = x + iy$ satisfies the equation $|z - 12 - 5i| = 3$, find the minimum value of $|z|$ and maximum value of $|z|$.

Solution

$|z-(12+5i)| = 3$

circle centre (12, 5)

$r = 3$



$|z|_{min} = OC - CX$

$= \sqrt{12^2 + 5^2} - 3$

$= 13 - 3$

$= 10$

$|z|_{max} = OC + CY$

$= 13 + 3$

$= 16$

Example

Given that $|z - 3| = |z + i|$,

a sketch the locus of z and find the Cartesian equation of this locus

b find the least possible value of $|z|$.

Solution

(a) $|z-3| = |z-(-i)|$

perp. bisector of $(3,0)$ & $(0,-1)$

grad of line joining $(3,0)$ and $(0,-1)$ $= \frac{0--1}{3-0} = \frac{1}{3}$

$\therefore$ perp grad $= -3$

$y = -3x + c$

Midpt of $(3,0)$ and $(0,-1)$

$= \left(\frac{3+0}{2}, \frac{0+-1}{2}\right)$

$= (1.5, -0.5)$

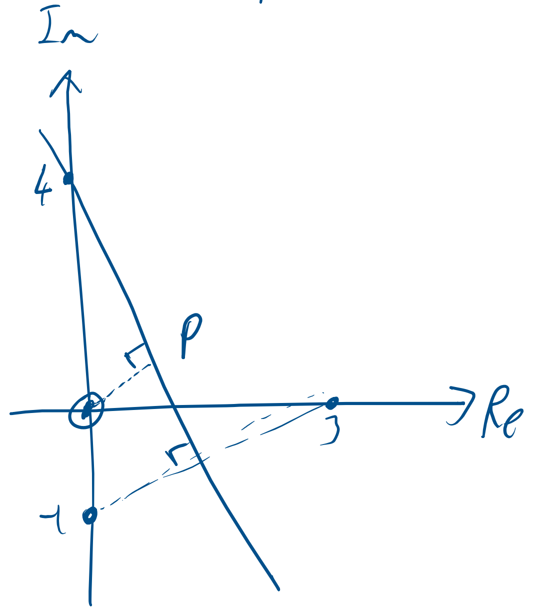
sub into $y = -3x + c$

$-0.5 = -3(1.5) + c$

$-0.5 = -4.5 + c$

$4 = c$

$\Rightarrow y = -3x + 4$



(a) OR

$|z-3| = |z+i|$

$|x+iy-3| = |x+iy+i|$

$|(x-3)+iy| = |x+i(y+1)|$

$\sqrt{(x-3)^2+y^2} = \sqrt{x^2+(y+1)^2}$

$x^2-6x+9+y^2 = x^2+y^2+2y+1$

$-6x+9-1 = 2y$

$2y = -6x+8$

$y = -3x+4$

(b) grad of line $OP = \frac{1}{3}$

$\therefore$ equation of OP is $y = \frac{1}{3}x$

find out where OP meets perp bisector $y = -3x+4$

$\frac{1}{3}x = -3x+4$

$3\frac{1}{3}x = 4$

$x = 4 \div 3\frac{1}{3}$

$x = \frac{6}{5}$ when $x = \frac{6}{5}$ $y = \frac{1}{3}\left(\frac{6}{5}\right)$

$= \frac{2}{5}$

$P\left(\frac{6}{5}, \frac{2}{5}\right)$

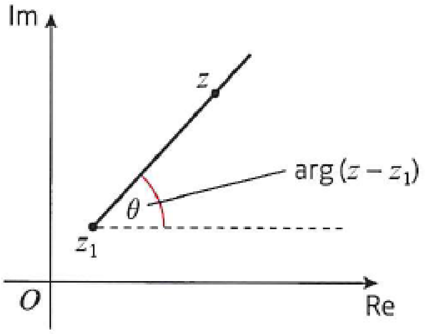
$\therefore$ dist $OP = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{2}{5}\right)^2}$

$= \frac{2\sqrt{10}}{5}$ units

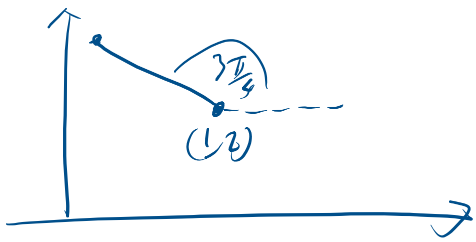
Locus questions can also make use of the geometric property of the argument.

- **Given $z_1 = x_1 + \mathrm{i}y_1$, the locus of points z on an Argand diagram such that $\arg(z - z_1) = \theta$ is a half-line from, but not including, the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.**

Notation A **half-line** is a straight line extending from a point infinitely in one direction only.



eg- $\arg(z-(1+2i))=\frac{3\pi}{4}$



You can find the Cartesian equation of the half-line corresponding to $\arg(z - z_1) = \theta$ by considering how the argument is calculated:

$$\arg(z - z_1) = \theta$$

$$\arg((x - x_1) + \mathrm{i}(y - y_1)) = \theta$$

$$\frac{y - y_1}{x - x_1} = \tan\theta$$

θ is a fixed angle so $\tan\theta$ is a constant.

$$y - y_1 = \tan\theta(x - x_1)$$

This is the equation of a straight line with gradient $\tan\theta$ passing through the point (x_1, y_1).

Example

Given that $\arg(z + 3 + 2\mathrm{i}) = \frac{3\pi}{4}$,

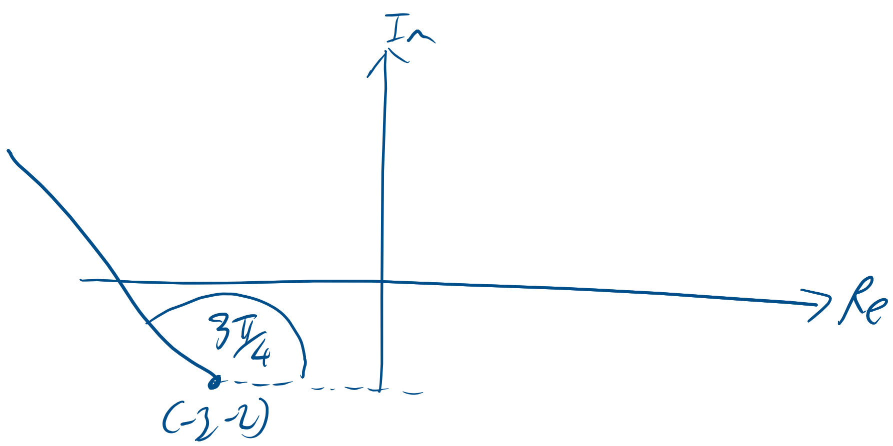
a sketch the locus of z on an Argand diagram

b find the Cartesian equation of the locus

c find the complex number z that satisfies both $|z + 3 + 2\mathrm{i}| = 10$ and $\arg(z + 3 + 2\mathrm{i}) = \frac{3\pi}{4}$

Solution

(a) $\arg(z-(-3-2i))=\frac{3\pi}{4}$



(b) line has gradient $\tan\theta$ ie, $\tan\frac{3\pi}{4} = -1$

and passes through $(-3,-2)$

$y = -x + c$

$(-3,-2)$ $\quad -2 = -1(-3) + c$

$-2 = 3 + c$

$-5 = c$

$\therefore y = -x - 5$

and $x < -3$.

OR $\arg(z + 3 + 2i) = \frac{3\pi}{4}$

$\arg((x+3) + i(y+2)) = \frac{3\pi}{4}$

$\frac{y+2}{x+3} = \tan\left(\frac{3\pi}{4}\right)$

$\frac{y+2}{x+3} = -1$

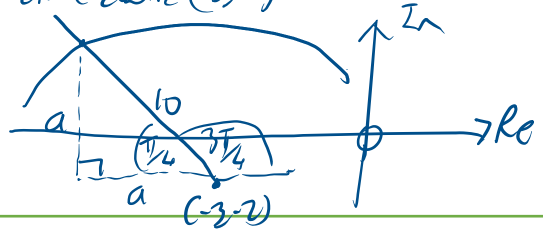
$y + 2 = -x - 3$

$y = -x - 5$ and $x < -3$

(c) $|z + 3 + 2i| = 10$

$|z - (-3-2i)| = 10$

is circle centre $(-3,-2)$ $r = 10$



By Pythagoras $a^2 + a^2 = 10^2$

$2a^2 = 100$

$a^2 = 50$

$a = 5\sqrt{2}$

$z = (-3 - 5\sqrt{2}) + i(-2 + 5\sqrt{2})$

OR sub equation $y = -x - 5$ into $(x+3)^2 + (y+2)^2 = 10^2$ and solve to get z value then choose appropriate one.

*Edexcel Book Ex2E Q1acei,4,6acei,7,10ace

*Complex Video 7

Regions in the Argand diagram

You can use complex numbers to represent regions on an Argand diagram.

Example

a On separate Argand diagrams, shade in the regions represented by:

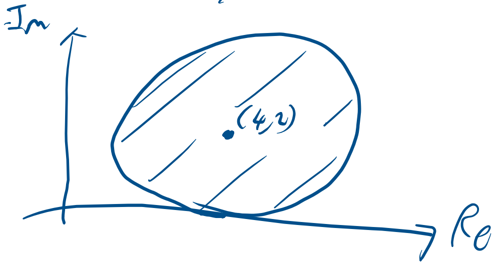
i $|z - 4 - 2i| \leqslant 2$ **ii** $|z - 4| < |z - 6|$ **iii** $0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}$

b Hence, on the same Argand diagram, shade the region which satisfies

$\{z \in \mathbb{C} : |z - 4 - 2i| \leqslant 2\} \cap \{z \in \mathbb{C} : |z - 4| < |z - 6|\} \cap \left\{z \in \mathbb{C} : 0 \leqslant \arg(z - 2 - 2i) \leqslant \frac{\pi}{4}\right\}$

Solution

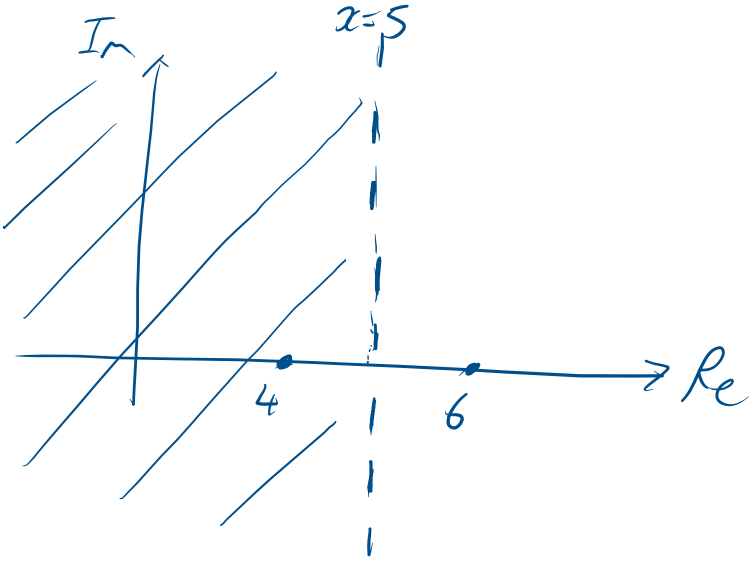
(a) (i) $|z - (4+2i)| \leqslant 2$



(ii) $|z-4| < |z-6|$

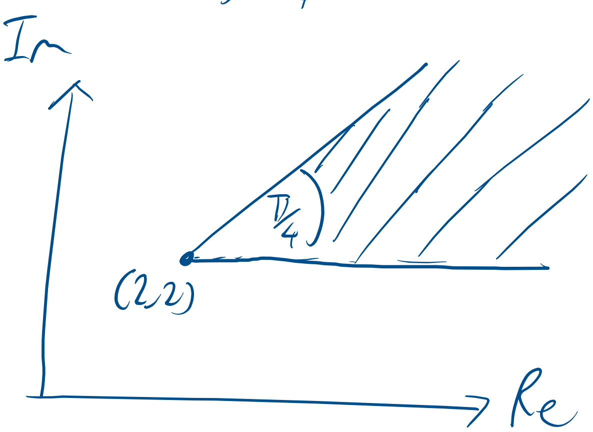
$*\ |z-4| = |z-6|$

is perp. bisector of $(4,0)$ & $(6,0)$



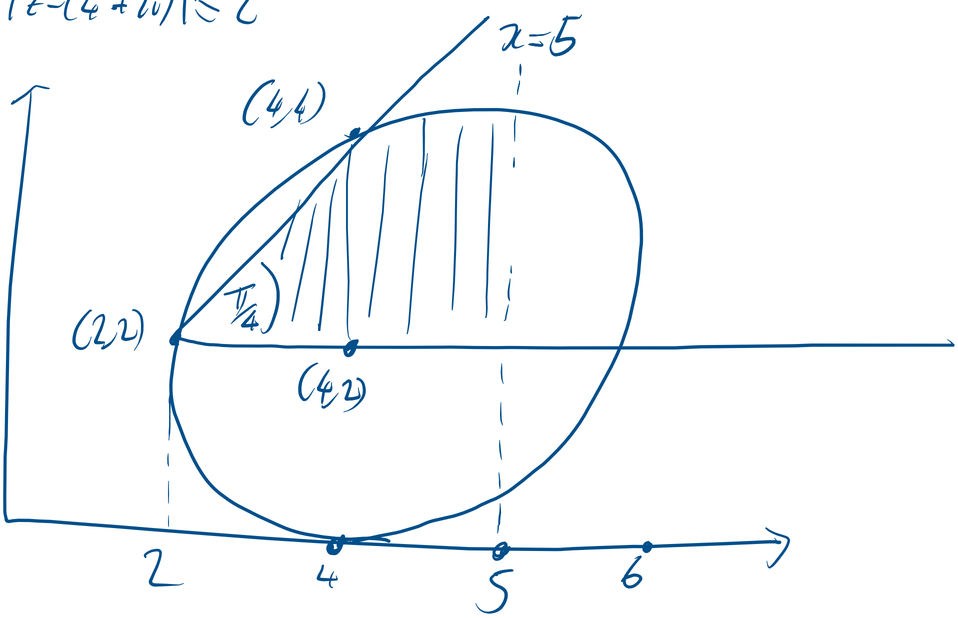
(iii) $0 \leq \arg(z-2-2i) \leq \frac{\pi}{4}$

$0 \leq \arg(z-(2+2i)) \leq \frac{\pi}{4}$



(c) $|z-4-2i| \leq 2$, $|z-4| < |z-6|$ and $0 \leq \arg(z-(2+2i)) \leq \frac{\pi}{4}$

$|z-(4+2i)| \leq 2$



Example

Sketch the locus of z such that

$$|z-3| = 2|z-1+i|$$

This means that the locus is the path of the point (x, y) such that the distance between $(3, 0)$ and (x, y) is twice the distance between $(1, -1)$ and (x, y). It is all very well to state this, but what sort of a curve does this give you? Since on this occasion the locus is not obvious, you need to resort to algebra. (You could have done this with the previous examples if you had got stuck.)

$$|z-3| = 2|z-1+i| \qquad {*}z = x+iy$$

$$|(x-3)+iy| = 2|(x-1) + (y+1)i|$$

$$\sqrt{(x-3)^2+y^2} = 2\sqrt{(x-1)^2+(y+1)^2}$$

$$(x-3)^2+y^2 = 4((x-1)^2+(y+1)^2)$$

$$x^2-6x+9+y^2 = 4(x^2-2x+1+y^2+2y+1)$$

$$x^2-6x-9+y^2 = 4(x^2-2x+y^2+2y+2)$$

$$x^2-6x-9+y^2 = 4x^2-8x+4y^2+8y+8$$

$$0 = 3x^2+3y^2-2x+8y-1 \qquad)\div 3$$

$$0 = x^2+y^2-\tfrac{2}{3}x+\tfrac{8}{3}y-\tfrac{1}{3}$$

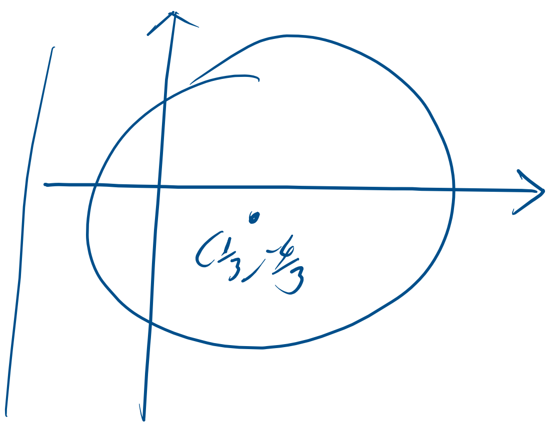
$$2g = -\tfrac{2}{3} \qquad 2f = \tfrac{8}{3} \qquad c = -\tfrac{1}{3}$$

$$g = -\tfrac{1}{3} \qquad f = \tfrac{4}{3} \qquad c = -\tfrac{1}{3}$$

centre $(-g, -f)$

$= (\tfrac{1}{3}, -\tfrac{4}{3})$

$$r = \sqrt{g^2+f^2-c} = \frac{\sqrt{20}}{3} = \frac{2\sqrt{5}}{3}$$



Example

Sketch the locus of z such that $\arg \dfrac{z-2}{z+5} = \dfrac{\pi}{4}$.

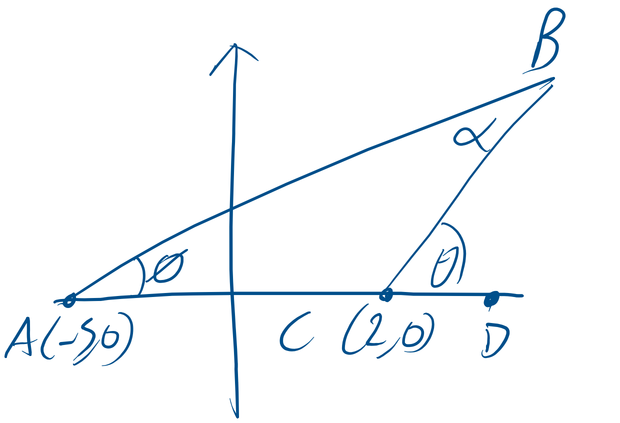
Solution

$z-2$ is represented by a vector from $(2,0)$ to (x,y)

$z+5$ is represented by a vector from $(-5,0)$ to (x,y)

Since $\arg\left(\frac{z-2}{z+5}\right) = \arg(z-2) - \arg(z+5)$

the equation $\arg\left(\frac{z-2}{z+5}\right) = \frac{\pi}{4}$ states that when you subtract the angle which vector $z+5$ makes with the positive x-axis from the angle that vector $z-2$ makes with positive x-axis you get $\frac{\pi}{4}$.



That is $\theta - \emptyset = \frac{\pi}{4}$

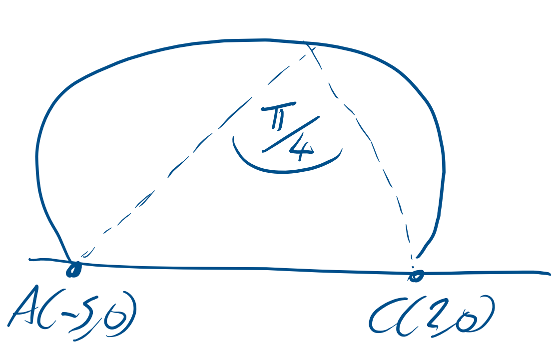
By Geometry you know that $\hat{ABC} + \hat{BAC} = \hat{BCD}$

ie. $\alpha + \emptyset = \theta$

$\therefore \alpha = \theta - \emptyset$

$\therefore \hat{ABC} = \frac{\pi}{4}$

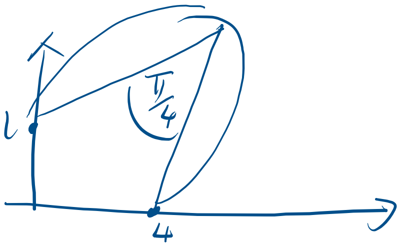
Again using Geometry you should recognise that B can vary such that $\hat{ABC} = \frac{\pi}{4}$, then B lies on the arc of a circle passing through A and C.



Result:

$\arg\left(\frac{z-a}{z-b}\right) = \lambda$ is an arc of a circle subtended from the points 'a' and 'b'.

eg. $\arg\left(\frac{z-i}{z-4}\right) = \frac{\pi}{4}$



*Edexcel Book Ex2F Q1aceg,2-6

* Edexcel Book Mixed Exercise 2 as extra

Summary of key points

1 You can represent complex numbers on an **Argand diagram**. The x-axis on an Argand diagram is called the **real axis** and the y-axis is called the **imaginary axis**. The complex number $z = x + \mathrm{i}y$ is represented on the diagram by the point $P(x, y)$, where x and y are Cartesian coordinates.

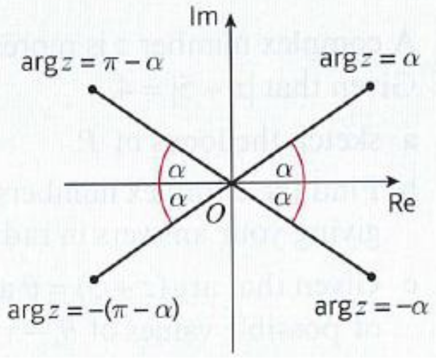
2 The complex number $z = x + \mathrm{i}y$ can be represented as the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ on an Argand diagram.

3 The **modulus** of a complex number, $|z|$, is the distance from the origin to that number on an Argand diagram. For a complex number $z = x + \mathrm{i}y$, the modulus is given by $|z| = \sqrt{x^2 + y^2}$.

4 The **argument** of a complex number, $\arg z$, is the angle between the positive real axis and the line joining that number to the origin on an Argand diagram. For a complex number $z = x + \mathrm{i}y$, the argument, θ, satisfies $\tan\theta = \frac{y}{x}$

5 Let α be the positive acute angle made with the real axis by the line joining the origin and z.

- If z lies in the first quadrant then $\arg z = \alpha$.
- If z lies in the second quadrant then $\arg z = \pi - \alpha$.
- If z lies in the third quadrant then $\arg z = -(\pi - \alpha)$.
- If z lies in the fourth quadrant then $\arg z = -\alpha$.



6 For a complex number z with $|z| = r$ and $\arg z = \theta$, the modulus–argument form of z is $z = r(\cos\theta + \mathrm{i}\sin\theta)$

7 For any two complex numbers z_1 and z_2,

- $|z_1 z_2| = |z_1||z_2|$
- $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

8 For two complex numbers $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, $|z_2 - z_1|$ represents the distance between the points z_1 and z_2 on an Argand diagram.

9 Given $z_1 = x_1 + \mathrm{i}y_1$, the locus of points z on an Argand diagram such that $|z - z_1| = r$, or $|z - (x_1 + \mathrm{i}y_1)| = r$, is a circle with centre (x_1, y_1) and radius r.

10 Given $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$, the locus of points z on an Argand diagram such that $|z - z_1| = |z - z_2|$ is the perpendicular bisector of the line segment joining z_1 and z_2.

11 Given $z_1 = x_1 + \mathrm{i}y_1$, the locus of points z on an Argand diagram such that $\arg(z - z_1) = \theta$ is a half-line from, but not including, the fixed point z_1 making an angle θ with a line from the fixed point z_1 parallel to the real axis.