Roots of Polynomials Questions

Exercise 4A

1 α and β are the roots of the quadratic equation $3x^2 + 7x - 4 = 0$. Without solving the equation, find the values of:

a $\alpha + \beta$ b $\alpha\beta$ c $\frac{1}{\alpha} + \frac{1}{\beta}$ d $\alpha^2 + \beta^2$

2 α and β are the roots of the quadratic equation $7x^2 - 3x + 1 = 0$. Without solving the equation, find the values of:

a $\alpha + \beta$ b $\alpha\beta$ c $\frac{1}{\alpha} + \frac{1}{\beta}$ d $\alpha^2 + \beta^2$

3 α and β are the roots of the quadratic equation $6x^2 - 9x + 2 = 0$. Without solving the equation, find the values of:

a $\alpha + \beta$ b $\alpha^2 \times \beta^2$

c $\frac{1}{\alpha} + \frac{1}{\beta}$ d $\alpha^3 + \beta^3$

Hint Try expanding $(\alpha + \beta)^3$.

4 The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = 2$ and $\beta = -3$. Find integer values for a, b and c.

5 The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{1}{2}$ and $\beta = -\frac{1}{3}$ Find integer values for a, b and c.

6 The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = \frac{-1 + i}{2}$ and $\beta = \frac{-1 - i}{2}$ Find integer values for a, b and c.

7 One of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $\alpha = -1 - 4i$.

a Write down the other root, β.

b Given that $a = 1$, find the values of b and c.

(P) 8 Given that $kx^2 + (k - 3)x - 2 = 0$, find the value of k if the sum of the roots is 4.

(P) 9 The equation $nx^2 - (16 + n)x + 256 = 0$ has real roots α and $-\alpha$. Find the value of n.

(P) 10 The roots of the equation $6x^2 + 36x + k = 0$ are reciprocals of each other. Find the value of k.

(P) 11 The equation $mx^2 + 4x + 4m = 0$ has roots of the form k and $2k$. Find the values of m and k.

(P) 12 The equation $ax^2 + 8x + c = 0$, where a and c are real constants, has roots α and α^*.

a Given that $\text{Re}(\alpha) = 2$, find the value of a.

b Given that $\text{Im}(\alpha) = 3i$, find the value of c.

(P) 13 The equation $4x^2 + px + q = 0$, where p and q are real constants, has roots α and α^*.

a Given that $\text{Re}(\alpha) = -3$, find the value of p.

b Given that $\text{Im}(\alpha) \neq 0$, find the range of possible values of q.

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Exercise 4A

1 a $-\frac{7}{3}$ b $-\frac{4}{3}$ c $\frac{7}{4}$ d $\frac{73}{9}$

2 a $\frac{3}{7}$ b $\frac{1}{7}$ c 3 d $-\frac{5}{49}$

3 a $\frac{3}{2}$ b $\frac{1}{9}$ c $\frac{9}{2}$ d $\frac{15}{8}$

4 $a = 1, b = 1, c = -6$

5 $a = 6, b = 5, c = 1$

6 $a = 2, b = 2, c = 1$

7 a $-1 + 4i$ b $b = 2, c = 17$

8 $\frac{3}{5}$

9 -16

10 6

11 $k = \sqrt{2}$ and $m = -\frac{2\sqrt{2}}{3}$ or $k = -\sqrt{2}$ and $m = \frac{2\sqrt{2}}{3}$

12 a -2 b -26

13 a 24 b $q > 36$

Exercise 4B

1 α, β and γ are the roots of the cubic equation $2x^3 + 5x^2 - 2x + 3 = 0$. Find the values of:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \beta\gamma + \gamma\alpha$ **d** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

2 α, β and γ are the roots of the cubic equation $x^3 + 5x^2 + 17x + 13 = 0$. Find the values of:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha\beta + \beta\gamma + \gamma\alpha$ **d** $\alpha^2\beta^2\gamma^2$

3 α, β and γ are the roots of the cubic equation $7x^3 - 4x^2 - x + 6 = 0$. Find the values of:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta\gamma$ **c** $\alpha^3\beta^3\gamma^3$ **d** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

4 The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$ and $\gamma = 1$.
Find integer values for a, b, c and d.

5 The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 + 3i$, $\beta = 1 - 3i$ and $\gamma = \frac{1}{2}$
Find integer values for a, b, c and d.

6 The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = \frac{5}{4}$, $\beta = -\frac{3}{2}$ and $\gamma = \frac{1}{2}$
Find integer values for a, b, c and d.

7 The cubic equation $16x^3 - kx^2 + 1 = 0$ has roots α, β and γ.

a Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **(2 marks)**

b i Given that $\alpha = \beta$, find the roots of the equation. **(5 marks)**

ii Find the value of k. **(1 mark)**

8 The cubic equation $2x^3 - kx^2 + 30x - 13 = 0$ has roots α, β and γ.

a Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$, and express k in terms of α, β and γ. **(3 marks)**

b Given that $\alpha = 2 - 3i$, find the value of k. **(4 marks)**

9 The cubic equation $x^3 - mx + n = 0$ has roots 1, −4 and α.

a State, with a reason, whether α is real. **(1 mark)**

b Find the values of m, n and α. **(4 marks)**

10 The cubic equation $2x^3 - 10x^2 + 8x - k = 0$ has a root at $x = 3 - i$.

a Find the other two roots of the equation. **(4 marks)**

b Hence find the value of k. **(2 marks)**

11 The cubic equation $x^3 - 14x^2 + 56x - 64 = 0$ has roots α, $k\alpha$ and $k^2\alpha$ for some real constant k.
Find the values of α and k. **(5 marks)**

12 Given that the roots of $8x^3 + 12x^2 - cx + d = 0$ are α, $\frac{\alpha}{2}$ and $\alpha - 4$, find α, c and d. **(5 marks)**

13 Given that the roots of the cubic equation $2x^3 + 48x^2 + cx + d = 0$ are α, 2α and 3α, find the values of α, c and d. **(5 marks)**

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Exercise 4B

1 **a** $-\frac{5}{2}$ **b** $-\frac{3}{2}$ **c** -1 **d** $\frac{2}{3}$
2 **a** -5 **b** -13 **c** 17 **d** 169
3 **a** $\frac{4}{7}$ **b** $-\frac{6}{7}$ **c** $-\frac{216}{343}$ **d** $\frac{1}{6}$
4 $a = 4, b = -12, c = 11, d = -3$
5 $a = 2, b = -5, c = 22, d = -10$
6 $a = 16, b = -4, c = -32, d = 15$
7 **a** $\alpha\beta + \beta\gamma + \gamma\alpha = 0, \alpha\beta\gamma = -\frac{1}{16}$
b i $\alpha = \frac{1}{2}, \beta = \frac{1}{2}, \gamma = -\frac{1}{4}$ **ii** 12
8 **a** $\alpha\beta + \beta\gamma + \gamma\alpha = 15, \alpha\beta\gamma = \frac{13}{2}, k = 2(\alpha + \beta + \gamma)$ **b** 9
9 **a** Yes – there are two other real roots, so α^* couldn't also be a root.
b $m = 13, n = 12, \alpha = 3$
10 **a** -1 and $3 + i$ **b** -20
11 $\alpha = 2$ and $k = 2$ or $\alpha = 8$ and $k = \frac{1}{2}$
12 $\alpha = 1, c = 32, d = 12$
13 $\alpha = -4, c = 352, d = 768$

Exercise 4C

1 α, β, γ and δ are the roots of the quartic equation $4x^4 + 3x^3 + 2x^2 - 5x - 4 = 0$. Without solving the equation, find the values of:

a $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

c $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$ **d** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

Hint $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma}{\alpha\beta\gamma\delta}$

2 α, β, γ and δ are the roots of the quartic equation $2x^4 + 4x^3 - 3x^2 - x + 2 = 0$. Find the values of:

a $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$ **c** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

d $\alpha\beta\gamma\delta$ **e** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$

3 α, β, γ and δ are the roots of the quartic equation $x^4 + 3x^3 + 2x^2 - x + 4 = 0$.
Find the values of:

a $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$ **c** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

d $\alpha\beta\gamma\delta$ **e** $\alpha^2\beta^2\gamma^2\delta^2$

4 α, β, γ and δ are the roots of the quartic equation $7x^4 + 6x^3 - 5x^2 + 4x + 3 = 0$.
Find the values of:

a $\alpha + \beta + \gamma + \delta$ **b** $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$ **c** $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta$

d $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta}$ **e** $\alpha^3\beta^3\gamma^3\delta^3$

5 The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are $\alpha = -\frac{3}{2}, \beta = -\frac{1}{2}, \gamma = -2$ and $\delta = \frac{2}{3}$
Find integer values for a, b, c, d and e.

6 The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are $\alpha = -\frac{1}{2}, \beta = \frac{1}{3}, \gamma = 1 + \text{i}$ and $\delta = 1 - \text{i}$. Find integer values for a, b, c, d and e.

7 The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are such that $\Sigma\alpha = \frac{17}{12}, \Sigma\alpha\beta = -\frac{25}{72}$, $\Sigma\alpha\beta\gamma = -\frac{53}{72}$ and $\alpha\beta\gamma\delta = -\frac{1}{6}$. Find integer values for a, b, c, d and e.

8 The quartic equation $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$ has roots $\alpha, \alpha + k, \alpha + 2k$ and $\alpha + 3k$ for some real constant k. Solve the equation. **(7 marks)**

9 The quartic equation $3072x^4 - 2880x^3 + 840x^2 - 90x + 3 = 0$ has roots $\alpha, r\alpha, r^2\alpha$ and $r^3\alpha$ for some real constant r. Solve the equation. **(7 marks)**

10 Three of the roots of the quartic equation $40x^4 + 90x^3 - 115x^2 + mx + n = 0$ are $1, -3$ and $\frac{1}{2}$

a Find the fourth root. **(2 marks)**

b Find the values of m and n. **(4 marks)**

11 The quartic equation $2x^4 - 34x^3 + 202x^2 + dx + e = 0$ has roots $\alpha, \alpha + 1, 2\alpha + 1$ and $3\alpha + 1$.

a Find α. **(2 marks)**

b Find the values of d and e. **(4 marks)**

12 The equation $4x^4 - 19x^3 + px^2 + qx + 10 = 0, x \in \mathbb{C}, p, q \in \mathbb{R}$, has roots α, β, γ and δ.
Given that $\gamma = 3 + \text{i}$ and $\delta = \gamma^*$,

a show that $4\alpha + 4\beta + 5 = 0$ and that $4\alpha\beta - 1 = 0$. **(2 marks)**

b Hence find all the roots of the quartic equation and find the values of p and q. **(5 marks)**

c Show these roots on an Argand diagram. **(3 marks)**

13 A quartic equation $6x^4 - 10x^3 + 3x^2 + 6x - 40 = 0$ has roots α, β, γ and δ.

a Show that $\frac{1 - 3\text{i}}{2}$ is one root of the equation. **(3 marks)**

b Without solving the equation, find the other roots. **(5 marks)**

c Show these roots on an Argand diagram. **(3 marks)**

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Exercise 4C

1 **a** $-\frac{3}{4}$ **b** $\frac{1}{2}$ **c** $\frac{5}{4}$ **d** $\frac{5}{4}$

2 **a** -2 **b** $-\frac{3}{2}$ **c** $\frac{1}{2}$ **d** 1 **e** $\frac{1}{2}$

3 **a** -3 **b** 2 **c** 1 **d** $\frac{1}{4}$ **e** 16

4 **a** $-\frac{6}{7}$ **b** $-\frac{5}{7}$ **c** $-\frac{4}{7}$ **d** $-\frac{4}{3}$ **e** $\frac{27}{343}$

5 $a = 12, b = 40, c = 25, d = -20, e = -12$

6 $a = 6, b = -11, c = 9, d = 4, e = -2$

7 $a = 72, b = -102, c = -25, d = 53, e = -12$

8 $x = 1, 3, 5$ or 7

9 $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ or $\frac{1}{16}$

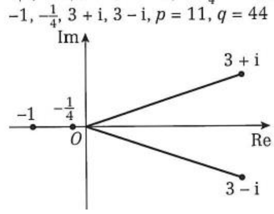
10 **a** $-\frac{3}{4}$ **b** $m = -60, n = 45$

11 **a** 2 **b** $d = -494, e = 420$

12 **a** $\alpha + \beta + (3 + \text{i}) + (3 - \text{i}) = \frac{19}{4} \Rightarrow 4\alpha + 4\beta + 5 = 0$
$\alpha\beta(3 + \text{i})(3 - \text{i}) = 10\alpha\beta = \frac{10}{4} \Rightarrow 4\alpha\beta - 1 = 0$

b $-1, -\frac{1}{4}, 3 + \text{i}, 3 - \text{i}, p = 11, q = 44$

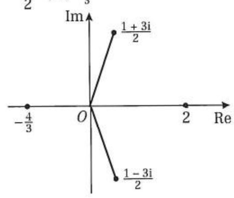
c



13 **a** $6\left(\frac{1 - 3\text{i}}{2}\right)^4 - 10\left(\frac{1 - 3\text{i}}{2}\right)^3 + 3\left(\frac{1 - 3\text{i}}{2}\right)^2 + 6\left(\frac{1 - 3\text{i}}{2}\right) - 40 = 0$

b $\frac{1 + 3\text{i}}{2}, 2, -\frac{4}{3}$

c



Exercise 4D

1 A quadratic equation has roots α and β. Given that $\alpha + \beta = 4$ and $\alpha\beta = 3$, find:

a $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **b** $\alpha^2\beta^2$ **c** $\alpha^2 + \beta^2$ **d** $\alpha^3 + \beta^3$

2 A quadratic equation has roots α and β. Given that $\alpha + \beta = -\frac{2}{3}$ and $\alpha\beta = \frac{3}{4}$, find:

a $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ **b** $\alpha^2\beta^2$ **c** $\alpha^2 + \beta^2$ **d** $\alpha^3 + \beta^3$

3 A quadratic equation has roots α and β. Given that $\alpha + \beta = \frac{5}{4}$ and $\alpha\beta = -\frac{1}{3}$, find:

a $(\alpha + 2)(\beta + 2)$ **b** $(\alpha - 4)(\beta - 4)$ **c** $(\alpha^2 + 1)(\beta^2 + 1)$

4 A cubic equation has roots α, β and γ. Given that $\alpha + \beta + \gamma = 2$, $\alpha\beta + \beta\gamma + \gamma\alpha = -3$ and $\alpha\beta\gamma = 4$, find:

a $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **b** $\alpha^2 + \beta^2 + \gamma^2$ **c** $\alpha^3 + \beta^3 + \gamma^3$ **d** $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$

5 A cubic equation has roots α, β and γ. Given that $\Sigma\alpha = \frac{3}{2}$, $\Sigma\alpha\beta = -\frac{4}{3}$ and $\alpha\beta\gamma = \frac{1}{2}$, find:

a $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **b** $\alpha^2 + \beta^2 + \gamma^2$ **c** $\alpha^3 + \beta^3 + \gamma^3$ **d** $\alpha^3\beta^3\gamma^3$

6 A cubic equation has roots α, β and γ. Given that $\alpha + \beta + \gamma = -\frac{1}{2}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{3}{4}$ and $\alpha\beta\gamma = -\frac{2}{5}$, find:

a $(\alpha + 2)(\beta + 2)(\gamma + 2)$ **b** $(\alpha - 3)(\beta - 3)(\gamma - 3)$ **c** $(1 - \alpha)(1 - \beta)(1 - \gamma)$

d $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2$ **e** $(\alpha\beta)^3 + (\beta\gamma)^3 + (\gamma\alpha)^3$

7 A quartic equation has roots α, β, γ and δ. Given that $\alpha + \beta + \gamma + \delta = 3$, $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 5$, $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -4$ and $\alpha\beta\gamma\delta = -2$, find:

a $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$ **b** $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ **c** $\alpha^4\beta^4\gamma^4\delta^4$

8 A quartic equation has roots α, β, γ and δ. Given that $\Sigma\alpha = \frac{1}{2}$, $\Sigma\alpha\beta = -\frac{3}{4}$, $\Sigma\alpha\beta\gamma = -\frac{1}{5}$ and $\alpha\beta\gamma\delta = \frac{4}{3}$, find:

a $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$ **b** $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ **c** $\alpha^3\beta^3\gamma^3\delta^3$

d $(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 + (\gamma\delta)^2 + (\alpha\delta)^2 + (\beta\delta)^2$

e $(\alpha\beta\gamma)^2 + (\alpha\beta\delta)^2 + (\alpha\gamma\delta)^2 + (\beta\gamma\delta)^2$

9 A quartic equation has roots α, β, γ and δ. Given that $\Sigma\alpha = -\frac{1}{2}$, $\Sigma\alpha\beta = -\frac{1}{3}$, $\Sigma\alpha\beta\gamma = \frac{1}{4}$ and $\alpha\beta\gamma\delta = \frac{3}{2}$, find:

a $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$ **b** $(2 - \alpha)(2 - \beta)(2 - \gamma)(2 - \delta)$

10 The roots of the equation $x^3 - 6x^2 + 9x - 15 = 0$ are α, β and γ.

a Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **(1 mark)**

b Hence find the values of:

i $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **(2 marks)**

ii $\alpha^2 + \beta^2 + \gamma^2$ **(2 marks)**

iii $(\alpha - 1)(\beta - 1)(\gamma - 1)$ **(3 marks)**

11 The roots of the equation $2x^3 + 4x^2 + 7 = 0$ are α, β and γ.

a Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **(1 mark)**

b Hence find the values of:

i $\alpha^2 + \beta^2 + \gamma^2$ **(2 marks)**

ii $\alpha^3\beta^3\gamma^3$ **(2 marks)**

iii $(\alpha + 2)(\beta + 2)(\gamma + 2)$ **(3 marks)**

12 Show that $\alpha^3 + \beta^3 + \gamma^3 \equiv (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$.

13 The roots of the equation $3x^3 - px + 11 = 0$ are α, β and γ.

a Given that $\alpha\beta + \beta\gamma + \gamma\alpha = 4$, write down the value of p. **(1 mark)**

b Write down the values of $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$. **(1 mark)**

c Hence find the value of $(3 - \alpha)(3 - \beta)(3 - \gamma)$. **(3 marks)**

14 The roots of the equation $x^4 + 2x^2 - x + 3 = 0$ are α, β, γ and δ.

a Write down the values of $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$ and $\alpha\beta\gamma\delta$. **(1 mark)**

b Hence find the values of:

i $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$ **(3 marks)**

ii $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ **(3 marks)**

iii $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$ **(3 marks)**

15 The roots of the equation $ax^4 + 3x^3 + 2x^2 + x - 6 = 0$ are α, β, γ and δ.

a Given that $\alpha\beta\gamma\delta = -3$, write down the value of a. **(1 mark)**

b Write down the values of $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\Sigma\alpha\beta\gamma$. **(1 mark)**

c Hence find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} + \dfrac{1}{\delta}$ **(3 marks)**

16 Prove that if a quartic equation has roots α, β, γ and δ then $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 \equiv (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$.

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Exercise 4D

1 a $\frac{4}{3}$ b 9 c 10 d 28

2 a $-\frac{8}{9}$ b $\frac{9}{16}$ c $-\frac{19}{18}$ d $\frac{65}{54}$

3 a $\frac{37}{6}$ b $\frac{32}{3}$ c $\frac{481}{144}$

4 a $-\frac{3}{4}$ b 10 c 38 d -7

5 a $-\frac{8}{3}$ b $\frac{59}{12}$ c $\frac{87}{8}$ d $\frac{1}{8}$

6 a $\frac{71}{10}$ b $-\frac{683}{20}$ c $\frac{53}{20}$ d $\frac{13}{80}$ e $\frac{723}{1600}$

7 a 2 b -1 c 16

8 a $-\frac{3}{20}$ b $\frac{7}{4}$ c $\frac{64}{27}$ d $\frac{823}{240}$ e $\frac{51}{25}$

9 a $\frac{23}{12}$ b $\frac{59}{3}$

10 a $\alpha + \beta + \gamma = 6$, $\alpha\beta + \beta\gamma + \gamma\alpha = 9$, $\alpha\beta\gamma = 15$

b i $\frac{3}{5}$ ii 18 iii 11

11 a $\alpha + \beta + \gamma = -2$, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$, $\alpha\beta\gamma = -\frac{7}{2}$

b i 4 ii $-\frac{343}{8}$ iii $-\frac{7}{2}$

12
$$(\alpha + \beta + \gamma)^3 \equiv (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha))$$
$$\equiv \alpha^3 + \beta^3 + \gamma^3 + \alpha(\beta^2 + \gamma^2) + \beta(\alpha^2 + \gamma^2) + \gamma(\alpha^2 + \beta^2) + 2(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$\equiv \alpha^2\beta + \beta^2\alpha + \alpha^2\gamma + \gamma^2\alpha + \beta^2\gamma + \gamma^2\beta + 3\alpha\beta\gamma$$
$$\equiv \alpha(\beta^2 + \gamma^2) + \beta(\alpha^2 + \gamma^2) + \gamma(\alpha^2 + \beta^2) + 3\alpha\beta\gamma$$
$$(\alpha + \beta + \gamma)^3 \equiv \alpha^3 + \beta^3 + \gamma^3 + 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma$$
$$\alpha^3 + \beta^3 + \gamma^3 \equiv (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$$

13 a -12 b $\alpha + \beta + \gamma = 0$, $\alpha\beta\gamma = -\frac{11}{3}$ c $\frac{128}{3}$

14 a $\sum\alpha = 0$, $\sum\alpha\beta = 2$, $\sum\alpha\beta\gamma = 1$, $\alpha\beta\gamma\delta = 3$

b i $\frac{1}{3}$ ii -4 iii 7

15 a 2 b $\sum\alpha = -\frac{3}{2}$, $\sum\alpha\beta = 1$, $\sum\alpha\beta\gamma = -\frac{1}{2}$ c $\frac{1}{6}$

16
$$\left(\sum\alpha\right)^2 \equiv (\alpha + \beta + \gamma + \delta)^2$$
$$\equiv \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + 2(\alpha\beta + \beta\gamma + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta)$$
$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \equiv \left(\sum\alpha\right)^2 - 2(\alpha\beta + \beta\gamma + \gamma\delta + \alpha\gamma + \beta\delta + \alpha\delta)$$

Exercise 4E

1 The cubic equation $x^3 - 7x^2 + 6x + 5 = 0$ has roots α, β and γ.
Find equations with roots:

a $(\alpha + 1)$, $(\beta + 1)$ and $(\gamma + 1)$ **b** 2α, 2β and 2γ

2 The cubic equation $3x^3 - 4x^2 - 5x + 1 = 0$ has roots α, β and γ.
Find equations with roots:

a $(\alpha - 3)$, $(\beta - 3)$ and $(\gamma - 3)$ **b** $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$

3 The cubic equation $x^3 - 3x^2 + 4x - 7 = 0$ has roots α, β and γ.
Without solving the equation, find the equation with roots $(2\alpha + 1)$, $(2\beta + 1)$ and $(2\gamma + 1)$.
Give your answer in the form $aw^3 + bw^2 + cw + d = 0$ where a, b, c and d are integers to be determined. **(5 marks)**

4 The cubic equation $x^3 + 4x^2 - 4x + 2 = 0$ has roots α, β and γ.
Without solving the equation, find the equation with roots $(2\alpha - 1)$, $(2\beta - 1)$ and $(2\gamma - 1)$.
Give your answer in the form $w^3 + pw^2 + qw + r = 0$ where p, q and r are integers to be found. **(5 marks)**

5 The cubic equation $3x^3 - x^2 + 2x - 5 = 0$ has roots α, β and γ.
Without solving the equation, find the equation with roots $(3\alpha + 1)$, $(3\beta + 1)$ and $(3\gamma + 1)$.
Give your answer in the form $aw^3 + bw^2 + cw + d = 0$ where a, b, c and d are integers to be determined. **(5 marks)**

6 The quartic equation $2x^4 + 4x^3 - 5x^2 + 2x - 1 = 0$ has roots α, β, γ and δ. Find equations with integer coefficients that have roots:

a 3α, 3β, 3γ and 3δ **b** $(\alpha - 1)$, $(\beta - 1)$, $(\gamma - 1)$ and $(\delta - 1)$

7 The quartic equation $x^4 + 2x^3 - 3x^2 + 4x + 5 = 0$ has roots α, β, γ and δ.
Without solving the equation, find equations with integer coefficients that have roots:

a 2α, 2β, 2γ and 2δ **(6 marks)**

b $(\alpha - 2)$, $(\beta - 2)$, $(\gamma - 2)$ and $(\delta - 2)$ **(6 marks)**

8 The quartic equation $3x^4 + 5x^3 - 4x^2 - 3x + 1 = 0$ has roots α, β, γ and δ.
Without solving the equation, find equations with integer coefficients that have roots:

a 3α, 3β, 3γ and 3δ **(6 marks)**

b $(\alpha + 1)$, $(\beta + 1)$, $(\gamma + 1)$ and $(\delta + 1)$ **(6 marks)**

ANSWERS

Exercise 4E

1 **a** $w^3 - 10w^2 + 23w - 9 = 0$
b $w^3 - 14w^2 + 24w + 40 = 0$

2 **a** $3w^3 + 23w^2 + 52w + 31 = 0$
b $24w^3 - 16w^2 - 10w + 1 = 0$

3 $w^3 - 9w^2 + 31w - 79 = 0$

4 $w^3 + 11w^2 + 3w + 9 = 0$

5 $w^3 - 4w^2 + 11w - 53 = 0$

6 **a** $2w^4 + 12w^3 - 45w^2 + 54w - 81 = 0$
b $2w^4 + 12w^3 + 19w^2 + 12w + 2=0$

7 **a** $w^4 + 4w^3 - 12w^2 + 32w + 80 = 0$
b $w^4 + 10w^3 + 33w^2 + 48w + 33 = 0$

8 **a** $w^4 + 5w^3 - 12w^2 - 27w + 27 = 0$
b $3w^4 - 7w^3 - w^2 + 8w - 2 = 0$

Mixed exercise 4

1 The roots of a quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ are $\alpha = \frac{1}{5}$, $\beta = -\frac{2}{5}$, $\gamma = -\frac{3}{5}$ and $\delta = -\frac{1}{2}$
Find integer values for a, b, c, d and e.

2 The cubic equation $x^3 + px^2 + 37x - 52 = 0$ has roots α, β and γ.

a Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$, and express p in terms of α, β and γ. **(3 marks)**

b Given that $\alpha = 3 - 2\text{i}$, find the value of p. **(4 marks)**

3 The cubic equation $2x^3 + 5x^2 - 2x + q = 0$ has a root at $x = -2 + \text{i}$.

a Find the other two roots of the equation. **(4 marks)**

b Hence find the value of q. **(2 marks)**

4 The quartic equation $x^4 - 40x^3 + 510x^2 - 2200x + 1729 = 0$ has roots α, $\alpha + 2k$, $\alpha + 4k$ and $\alpha + 6k$ for some real constant k. Solve the equation. **(7 marks)**

5 Three of the roots of the quartic equation $24x^4 - 58x^3 + 17x^2 + dx + e = 0$ are $\frac{1}{2}$, $-\frac{1}{3}$ and 2.

a Find the fourth root. **(2 marks)**

b Find the values of d and e. **(4 marks)**

6 The equation $x^4 + 2x^3 + mx^2 + nx + 85 = 0$, $x \in \mathbb{C}$, $m, n \in \mathbb{R}$, has roots α, β, γ and δ.
Given that $\alpha = -2 + \text{i}$ and $\beta = \alpha^*$,

a show that $\gamma + \delta - 2 = 0$ and that $\gamma\delta - 17 = 0$. **(2 marks)**

b Hence find all the roots of the quartic equation and find the values of m and n. **(5 marks)**

c Show these roots on an Argand diagram. **(3 marks)**

7 A quartic equation $4x^4 - 16x^3 + 115x^2 + 4x - 29 = 0$ has roots α, β, γ and δ.

a Show that $2 - 5\text{i}$ is one root of the equation. **(3 marks)**

b Without solving the equation, find the other roots. **(5 marks)**

c Show these roots on an Argand diagram. **(3 marks)**

8 The roots of the equation $2x^3 - 5x^2 + 11x - 9 = 0$ are α, β and γ.

a Write down the values of $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **(1 mark)**

b Hence find the values of:

i $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **(2 marks)**

ii $\alpha^2 + \beta^2 + \gamma^2$ **(2 marks)**

iii $(\alpha - 1)(\beta - 1)(\gamma - 1)$ **(3 marks)**

9 The roots of the equation $px^4 + 12x^3 + 6x^2 + 5x - 7 = 0$ are α, β, γ and δ.

a Given that $\alpha\beta\gamma\delta = -1$, write down the value of p. **(1 mark)**

b Write down the values of $\Sigma\alpha$, $\Sigma\alpha\beta$ and $\Sigma\alpha\beta\gamma$. **(1 mark)**

c Hence find the value of $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. **(3 marks)**

10 The roots of the equation $5x^3 + cx + 21 = 0$ are α, β and γ.

a Given that $\alpha\beta + \beta\gamma + \gamma\alpha = -6$, write down the value of c. **(1 mark)**

b Write down values for $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$. **(1 mark)**

c Hence find the value of $(1 - \alpha)(1 - \beta)(1 - \gamma)$. **(3 marks)**

11 The cubic equation $2x^3 + 5x^2 + 7x - 2 = 0$ has roots α, β and γ.
Without solving the equation, find the equation with roots $(3\alpha + 1)$, $(3\beta + 1)$ and $(3\gamma + 1)$.
Give your answer in the form $pw^3 + qw^2 + rw + s = 0$ where p, q, r and s are integers to be found. **(5 marks)**

12 The quartic equation $6x^4 - 2x^3 - 5x^2 + 7x + 8 = 0$ has roots α, β, γ and δ.
Without solving the equation, find equations with integer coefficients that have roots:

a 2α, 2β, 2γ and 2δ **(6 marks)**

b $(3\alpha - 2)$, $(3\beta - 2)$, $(3\gamma - 2)$ and $(3\delta - 2)$ **(6 marks)**

ANSWERS

Mixed exercise 4

1 $a = 250$, $b = 325$, $c = 110$, $d = -7$, $e = -6$

2 **a** $\alpha\beta + \beta\gamma + \gamma\alpha = 37$, $\alpha\beta\gamma = 52$, $p = -\alpha - \beta - \gamma$

b -10

3 **a** $-2 - \mathrm{i}, \frac{3}{2}$ **b** -15

4 $x = 1, 7, 13$ or 19

5 **a** $\frac{1}{4}$

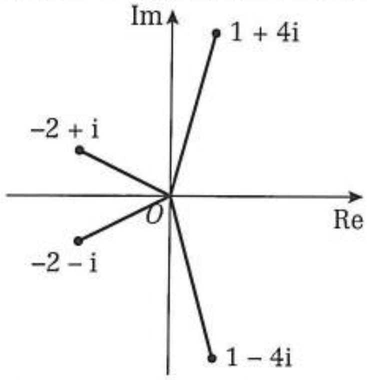
b $d = 7$, $e = -2$

6 **a** $(-2 + \mathrm{i}) + (-2 - \mathrm{i}) + \gamma + \delta = -2 \Rightarrow \gamma + \delta - 2 = 0$

$(-2 + \mathrm{i})(-2 - \mathrm{i})\gamma\delta = 85 \Rightarrow 5\gamma\delta = 85 \Rightarrow \gamma\delta - 17 = 0$

b Roots: $-2 \pm \mathrm{i}$, $1 \pm 4\mathrm{i}$; $m = 14$, $n = 58$

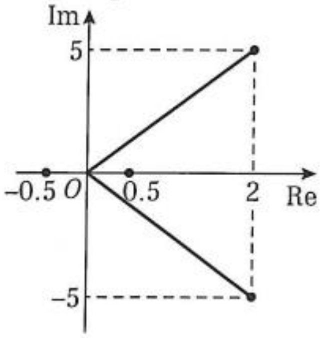
c



7 **a** $4(2 - 5\mathrm{i})^4 - 16(2 - 5\mathrm{i})^3 + 115(2 - 5\mathrm{i})^2 + 4(2 - 5\mathrm{i}) - 29 = 0$

b $2 + 5\mathrm{i}, \pm\frac{1}{2}$

c



8 **a** $\alpha + \beta + \gamma = \frac{5}{2}$, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{11}{2}$, $\alpha\beta\gamma = \frac{9}{2}$

b **i** $\frac{11}{9}$ **ii** $-\frac{19}{4}$ **iii** $\frac{1}{2}$

9 **a** 7 **b** $\sum\alpha = -\frac{12}{7}$, $\sum\alpha\beta = \frac{6}{7}$, $\sum\alpha\beta\gamma = -\frac{5}{7}$ **c** $\frac{60}{49}$

10 **a** -30 **b** $\alpha + \beta + \gamma = 0$, $\alpha\beta\gamma = -\frac{21}{5}$ **c** $-\frac{4}{5}$

11 $2w^3 + 9w^2 + 39w - 104 = 0$

12 **a** $3w^4 - 2w^3 - 10w^2 + 28w + 64 = 0$

b $2w^4 + 14w^3 + 21w^2 + 43w + 298 = 0$