Vectors

A scalar quantity is one which requires only size to describe it.

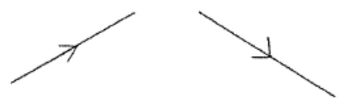
Temperature and distance can be given by a number of the appropriate unit of measurement (21°C, 35 cm).

Some quantities need a direction as well as a number of the appropriate unit of measurement, e.g. 35 km/h north.

A quantity that has both size and direction is called a vector.

Vectors can be represented on diagrams by lines known as "directed line segments".

A "directed line segment" is a line with an arrow:



The length of the line represents the size, the position of the line and the arrow represent the direction. These "directed line segments" have the same length but different directions so they represent different vectors.

Vector Notation

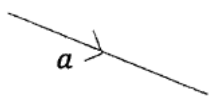
There are 2 common notations:

(i)

We would refer to this vector as: $\overrightarrow{AB}$

In this notation the arrow above AB is always shown from left to right.

(ii)



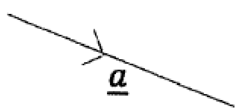
We would refer to this vector as: $\underline{a}$

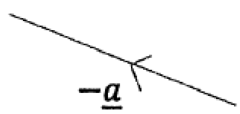
In text books and exams it will be shown in bold as $\boldsymbol{a}$.

In hand-written work you must underline EVERY LETTER representing a VECTOR. It will be marked WRONG in exams if you do not. DO NOT ATTEMPT TO WRITE THEM IN BOLD.

Two vectors are **equal** if they have the same length and are in the same direction.

A vector with the same length as vector $\boldsymbol{a}$ but the opposite direction is the vector $-\boldsymbol{a}$.





Multiplying a vector by a scalar

A scalar is an ordinary number. If the scalar is a positive number then the length of the vector increases but the direction stays the same.

If $\boldsymbol{b} = 2\boldsymbol{a}$ then we say that $\boldsymbol{b}$ is a scalar multiple of $\boldsymbol{a}$.

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same direction (they are parallel) but different lengths.

We can also see this if we write $\boldsymbol{a}$ and $\boldsymbol{b}$ in column vector form.

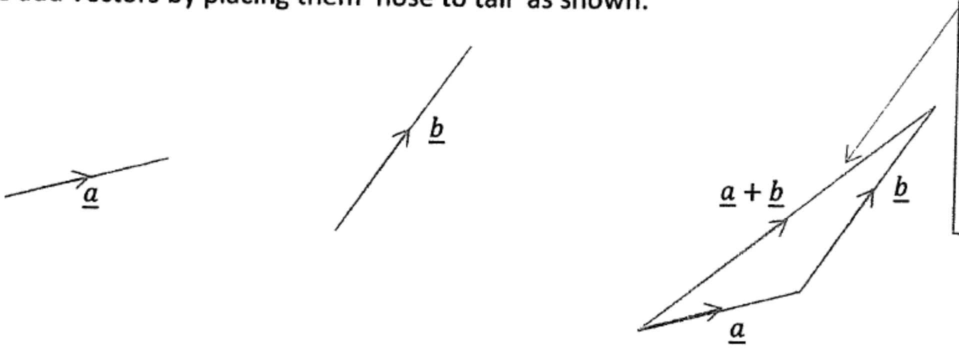
Example

If $\boldsymbol{a} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$, show that $\boldsymbol{b}$ is a scalar multiple of $\boldsymbol{a}$.

When vectors are in the same direction they are either parallel or in a straight line.

Addition of vectors

We add vectors by placing them 'nose to tail' as shown:



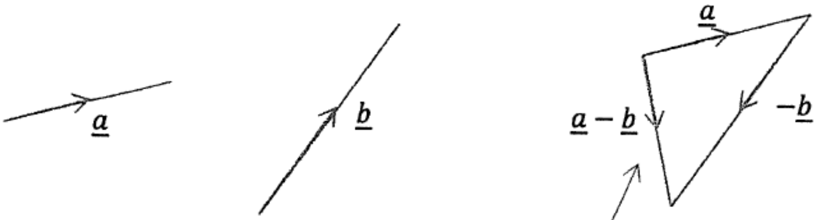
The vector in red is the sum of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Note how it starts at the beginning of $\boldsymbol{a}$ and finishes at the end of $\boldsymbol{b}$

Subtraction of vectors

We do this in a similar way to addition by thinking of 'subtraction' as 'addition of the negative'.

So $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$ gives the following:



The vector in red is the difference of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

When we add or subtract vectors the answer is called the **resultant vector**.

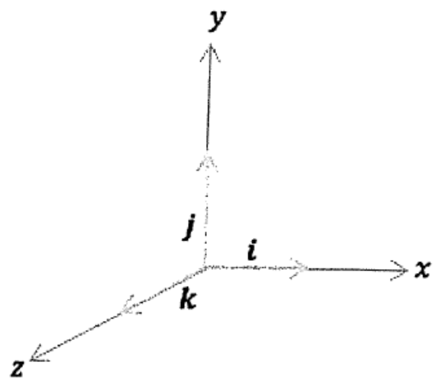
Alternative vector format

Using the standard Cartesian axes we denote:

A vector of length one in the direction of the positive x −axis as $\boldsymbol{i}$.

A vector of length one in the direction of the positive y −axis as $\boldsymbol{j}$.

A vector of length one in the direction of the positive z −axis as $\boldsymbol{k}$.



Every vector can now be represented in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ and these are called the **component vectors** in the x, y and z direction respectively.

Example

Given $\boldsymbol{a} = (3\boldsymbol{i} + 2\boldsymbol{j} + 8\boldsymbol{k})$ and $\boldsymbol{b} = (5\boldsymbol{i} - 6\boldsymbol{j} - 4\boldsymbol{k})$, find the resultant of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Column Vectors

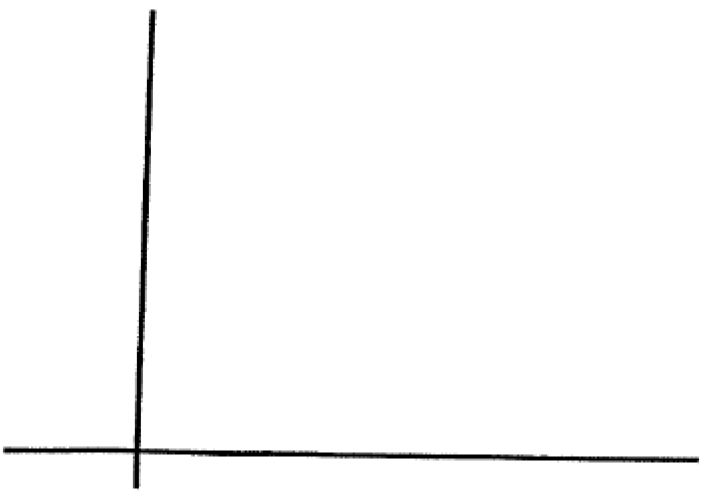
Column vectors are a quicker version of writing vectors. In the above example, we can write

$$\boldsymbol{a} = \begin{pmatrix} 3 \\ 2 \\ 8 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 5 \\ -6 \\ -4 \end{pmatrix} \qquad \therefore \boldsymbol{a} + \boldsymbol{b} =$$

Position Vectors

A vector $\boldsymbol{a}$ defining the position of a point A in relation to the origin O is called the **position vector** of point A i.e. $\boldsymbol{a} = \overrightarrow{OA}$

Consider point $P(3,2)$, then the position vector $\overrightarrow{OP}$ can be written as



Displacement Vectors

A vector representing movement from point A to point B is called a displacement vector.

Example

Point A has position vector $4\boldsymbol{i} + 3\boldsymbol{j}$ and B has position vector $-2\boldsymbol{i} + 4\boldsymbol{j}$. Find the displacement vector $\overrightarrow{AB}$.

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$ where **a** and **b** represent position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively

Magnitude of a Vector

The **magnitude** of a vector represents the vector's size. The magnitude of vector $\boldsymbol{a}$ is denoted by $|\boldsymbol{a}|$. If the vector $\overrightarrow{AB}$ is represented by the line AB then the magnitude is the distance from A to B.

E.g. If $\boldsymbol{a} = x\boldsymbol{i} + y\boldsymbol{j}$ and $\boldsymbol{b} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ then $|\boldsymbol{a}| = \sqrt{x^2 + y^2}$ and $|\boldsymbol{b}| = \sqrt{x^2 + y^2 + z^2}$

N.B. A **unit vector** is a vector of **magnitude 1**.

The unit vector in the direction of vector $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$. It is defined as $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}(\boldsymbol{a}) = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$

Example 1

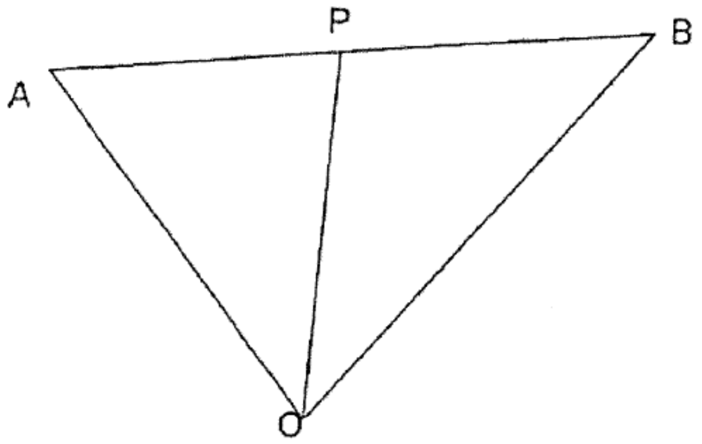
Find a unit vector in the direction of the vector $8\boldsymbol{i} - 6\boldsymbol{j}$.

Example 2

Find a vector of magnitude 14 in the direction of the vector $6\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$.

Ratio Theorem

The point P which divides the line AB in the ratio λ : μ has position vector **p** where $\frac{\mu \mathbf{a} + \lambda \mathbf{b}}{\lambda + \mu}$ where **a** and **b** are the position vectors of A and B respectively



Example

If a point A has position vector **i** + 2**j** and B has a position vector 5**i** + **j** find the position vector of the point which divides AB in the ration 1 : -3

(NB the negative sign means the point divides AB externally rather than internally)

Note

The unit vectors ***i*** and ***j*** are base vectors from which other co-planer vectors can be built up. Any pair of non-parallel co-planer vectors could be used instead.

Example

With $\boldsymbol{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ as base vectors, express $\boldsymbol{c} = \begin{pmatrix} 5 \\ 17 \end{pmatrix}$ in the form $\lambda\boldsymbol{a} + \mu\boldsymbol{b}$.

*UPM Ex2B Q3,4,6,7,9,10,11,13,18,19,20

The Scalar Product /Dot Product

The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as the product of the magnitudes of the two vectors multiplied by the cosine of the angle between the 2 vectors

$$\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

The scalar product is also referred to as the **dot product**. $|\boldsymbol{a}|, |\boldsymbol{b}|$ and $\cos\theta$ are scalar quantities.

The 'angle between' refers to the angle between the directions of the vectors where these directions are either both towards or both away from the point of intersection i.e.

Note that if θ is acute, the scalar product will be positive. If θ is obtuse, the scalar product will be negative.

The scalar product can also be calculated using components, i.e. if $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ then

$$\boldsymbol{a}.\boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

Hence the angle between the two vectors can be calculated:

$$\cos\theta = \frac{\boldsymbol{a}.\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

Notes

1. The scalar product of 2 perpendicular vectors is zero since $\cos 90 = 0$ i.e. $\boldsymbol{a}.\boldsymbol{b} = 0$.
2. $\boldsymbol{a}.\boldsymbol{b} = \boldsymbol{b}.\boldsymbol{a}$
3. $\boldsymbol{a}.\boldsymbol{a} = |\boldsymbol{a}||\boldsymbol{a}|\cos 0 = |\boldsymbol{a}|^2$
4. $(\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{c} = \boldsymbol{a}.\boldsymbol{c} + \boldsymbol{b}.\boldsymbol{c}$

Example 1

Find the angle between the two vectors $\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ and $2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}$.

Example 2

Given that two vectors $\boldsymbol{a}=(3t+1)\boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k}$ and $\boldsymbol{b}=(t+3)\boldsymbol{i}+3\boldsymbol{j}-2\boldsymbol{k}$ are perpendicular, find the possible values of t.

Example 3

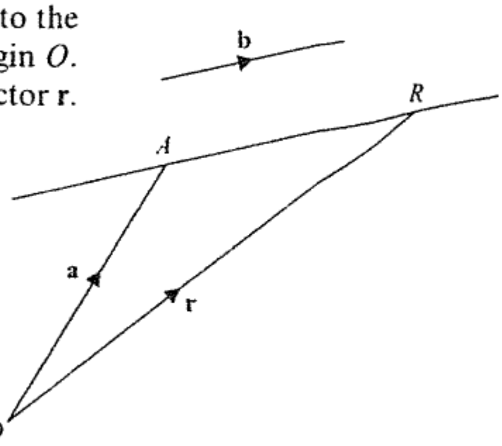
Show that ΔABC is a right angled triangle and find the other two angles given $A(5,3,2)$, $B(2,-1,3)$ and $C(7,-3,10)$.

<u>*UPM Ex2C Q1,3,5,6,8,11,12,15,16</u>

<u>*UPM Ex17A Q1,2,4,6,7,11,12</u>

The vector equation of a straight line

Think of a line that passes through the point A and is parallel to the vector $\mathbf{b}$. The point A has position vector $\mathbf{a}$ referred to the origin O. Let R be any other point on the line and let it have position vector $\mathbf{r}$. Since the line is parallel to $\mathbf{b}$, then:



$$\overrightarrow{AR} = \lambda\mathbf{b}, \text{ where } \lambda \text{ is a scalar}$$

However:

$$\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$$

So: $\quad \mathbf{r} - \mathbf{a} = \lambda\mathbf{b}$

or $\quad \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$

This is a vector equation of the straight line. The vector $\mathbf{b}$ is in the same direction as the line and is sometimes called the **direction vector of the line**. The vector $\mathbf{a}$ is the position vector of a point on the line and λ is a scalar taking all real values.

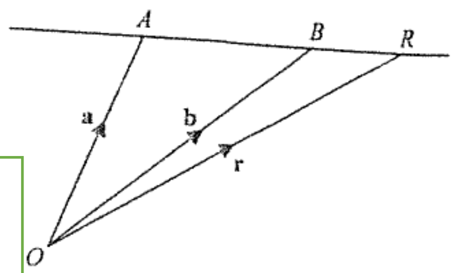
<u>Example</u>

Find a vector equation of the line that passes through the point with position vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and is parallel to the vector $-5\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

<u>Example</u>

Find a vector equation of the line that passes through the points A and B with position vectors $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ respectively.

Since the line passes through the points A and B a direction vector for the line is $\overrightarrow{AB}$. (Notice that $\overrightarrow{BA}$ is also a direction vector for the line.) Then:



Example Find the point of intersection of $L_1: \boldsymbol{r} = 2\boldsymbol{i} + \boldsymbol{j} + \lambda(\boldsymbol{i} + 3\boldsymbol{j})$ and $L_2: \boldsymbol{r} = 6\boldsymbol{i} - \boldsymbol{j} + \mu(\boldsymbol{i} - 4\boldsymbol{j})$.

Solution

Example Find the vector equation of the line that passes through the point with position vector $2\boldsymbol{i} + 3\boldsymbol{j}$ and is perpendicular to the line $\boldsymbol{r} = 3\boldsymbol{i} + 2\boldsymbol{j} + \lambda(\boldsymbol{i} - 2\boldsymbol{j})$.

Solution

Example

Find the perpendicular distance from the point A, position vector $\begin{pmatrix} 4 \\ -3 \\ 10 \end{pmatrix}$ to the line L, vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

Solution

Definition

Skew lines are lines in 3-D which are not parallel and do not intersect.

Example

Show that the following lines are skew.

line 1: $\mathbf{r} = 17\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} + \lambda(-9\mathbf{i} + 3\mathbf{j} + 9\mathbf{k})$
line 2: $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \mu(6\mathbf{i} + 7\mathbf{j} - \mathbf{k})$

Solution

Cartesian equation of a line in three dimensions

Consider a line that is parallel to the vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ and which passes through the point A, position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. If the general point on this line has position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then the vector equation of the line is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda\begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Thus $x = a + \lambda p$, $y = b + \lambda q$ and $z = c + \lambda r$ — These are the parametric equations of the line, using the parameter λ.

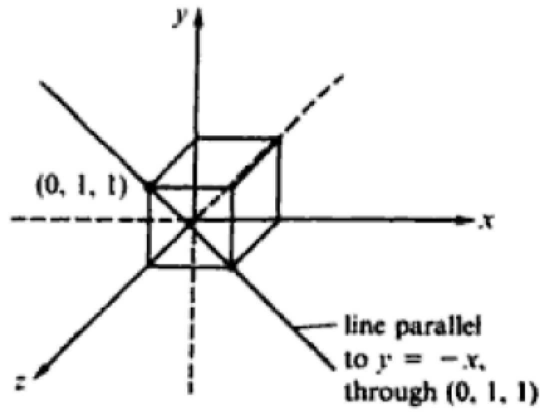
Isolating λ in each equation gives $\dfrac{x-a}{p} = \dfrac{y-b}{q} = \dfrac{z-c}{r} (= \lambda)$. These are the cartesian equations of the line

> Thus the line with vector equation $\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ has cartesian equations: $\dfrac{x-a}{p} = \dfrac{y-b}{q} = \dfrac{z-c}{r}$

Notes

1. Given the cartesian equation of a line in the above form it is easy to obtain the vector equation by remembering that the numerator gives the position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ of a point on the line and the denominator gives the direction vector $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$.
2. It is acceptable to give the cartesian equation of a line in the above form even when one or more of p, q and r are zero.

   For example, the line through (0, 1, 1) and parallel to $y = -x$, i.e. parallel to the vector $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, has vector equation $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, (see diagram).



   This gives the parametric equations $\begin{cases} x = -\lambda \\ y = 1 + \lambda \\ z = 1 \end{cases}$

   The cartesian equations of the line can then be written $\dfrac{x}{-1} = \dfrac{y-1}{1} = \dfrac{z-1}{0}$ or simply as $\dfrac{x}{-1} = \dfrac{y-1}{1}$, $z = 1$.

Example

Find the cartesian equations of the line that is parallel to the vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and which passes through the point A, position vector $3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

Solution

The vector equation of the line is $\mathbf{r} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

thus

$$x = 3 + 2\lambda$$
$$y = -1 + 3\lambda$$
$$z = 2 + 4\lambda$$

The cartesian equations are therefore $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-2}{4} (= \lambda)$

Example

The lines L_1 and L_2 are $\frac{x}{1} = \frac{y+2}{2} = \frac{z-5}{-1}$ and $\frac{x-1}{-1} = \frac{y+3}{-3} = \frac{z-4}{1}$. Show that L_1 and L_2 intersect and find the point of intersection.

Solution

*UPM Ex17C Q1-4,6,7,9a,10-13,15-17,19,20ac

The vector product

The scalar product of the vectors **a** and **b** is

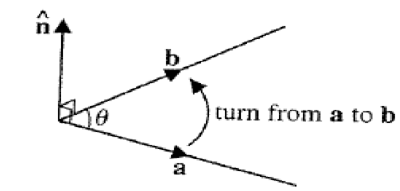
$$\mathbf{a}.\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

where θ is the angle between the vectors **a** and **b**

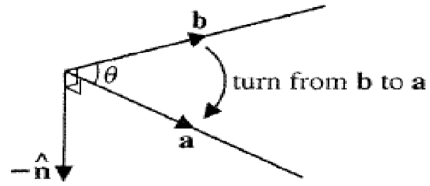
The **vector** (or **cross**) **product** of the vectors **a** and **b** is defined as

■ $$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\hat{\mathbf{n}}$$

Once again θ is the angle between **a** and **b**, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both **a** and **b**. The direction of $\hat{\mathbf{n}}$ is that in which a right-handed corkscrew would move when turned from **a** to **b**:



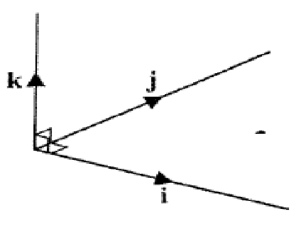
If the turn is in the opposite sense, that is from **b** to **a**, then the movement of the corkscrew is in the opposite sense to $\hat{\mathbf{n}}$. That is, it is in the direction of $-\hat{\mathbf{n}}$.



So:
$$\begin{aligned}\mathbf{b} \times \mathbf{a} &= |\mathbf{b}|\,|\mathbf{a}|\,\sin\theta(-\hat{\mathbf{n}})\\ &= -|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta\,\hat{\mathbf{n}}\\ &= -\mathbf{a} \times \mathbf{b}\end{aligned}$$

■ $$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Be careful, therefore, because $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$. The vector product is *not* commutative. Notice that $\mathbf{a}.\mathbf{b}$ is called the scalar product of **a** and **b** because the result is a scalar and that $\mathbf{a} \times \mathbf{b}$ is called the vector product of **a** and **b** because the result is another vector.



Vector product equal to zero

If $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then since $\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \sin\theta\, \hat{\mathbf{n}}$

either $\quad |\mathbf{a}| = 0 \Rightarrow \mathbf{a} = \mathbf{0}$

or $\quad |\mathbf{b}| = 0 \Rightarrow \mathbf{b} = \mathbf{0}$

or $\quad \sin\theta = 0 \Rightarrow \theta = 0 \text{ or } \pi$

But if either $\theta = 0$ or $\theta = \pi$ then $\mathbf{a}$ and $\mathbf{b}$ are in the same direction (and either in the same sense or in opposite senses). In either case, if $\sin\theta = 0$, then $\mathbf{a}$ and $\mathbf{b}$ are parallel.

So:

- if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$ then either $\mathbf{a} = \mathbf{0}$, or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel.

Now the vectors **i**, **j** and **k** are such that each one is perpendicular to the other two. Also their relative positions are such that if a right-handed corkscrew were turned from **i** to **j** it would move in the direction **k**. You should also be able to see that if a right-handed corkscrew were turned from **j** to **k** it would move in the direction **i** and if it were turned from **k** to **i**, it would move in the direction **j**.

So $\quad \mathbf{i} \times \mathbf{j} = |\mathbf{i}|\,|\mathbf{j}| \sin 90^\circ\, \mathbf{k}$

$\quad = (1 \times 1 \times 1)\mathbf{k} = \mathbf{k}$

- That is: $\quad \mathbf{i} \times \mathbf{j} = \mathbf{k}$

  Similarly $\quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$

  and $\quad \mathbf{k} \times \mathbf{i} = \mathbf{j}$

and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

Also, since the angle between **i** and itself is 0 and $\sin 0 = 0$,

- then: $\quad \mathbf{i} \times \mathbf{i} = \mathbf{0}$

  Similarly $\quad \mathbf{j} \times \mathbf{j} = \mathbf{0}$

  and $\quad \mathbf{k} \times \mathbf{k} = \mathbf{0}$

The vector product of $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$

$= a_1b_1(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + a_1b_3(\mathbf{i} \times \mathbf{k}) + a_2b_1(\mathbf{j} \times \mathbf{i})$
$\quad + a_2b_2(\mathbf{j} \times \mathbf{j}) + a_2b_3(\mathbf{j} \times \mathbf{k}) + a_3b_1(\mathbf{k} \times \mathbf{i}) + a_3b_2(\mathbf{k} \times \mathbf{j}) + a_3b_3(\mathbf{k} \times \mathbf{k})$

$= a_1b_2\mathbf{k} + a_1b_3(-\mathbf{j}) + a_2b_1(-\mathbf{k}) + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} + a_3b_2(-\mathbf{i})$

(because $\mathbf{i} \times \mathbf{k} = -\mathbf{k} \times \mathbf{i} = -\mathbf{j}$, $\mathbf{j} \times \mathbf{i} = -\mathbf{i} \times \mathbf{j} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{j} \times \mathbf{k} = -\mathbf{i}$)

$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Result

- $$(a_1\mathbf{i} + +a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example

Find a unit vector which is perpendicular to both $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

Solution

Example

Find the sine of the acute angle between $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$.

Solution

*P4 Book page 213 EX6A Q1alt,2,4,5,6,7alt,9,10

Applications of the vector product

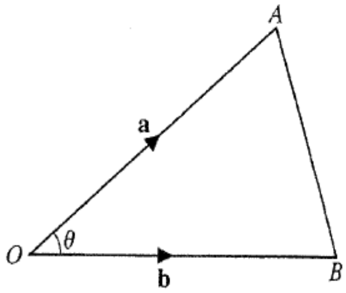
Area of a triangle

You should know that if you are given a triangle ABC then the area of the triangle is given by $\frac{1}{2}ab \sin C$.

If, therefore, you have a triangle OAB, then the area of the triangle is given by

$$\tfrac{1}{2}(OA)(OB)\sin\theta$$

where $\angle AOB = \theta$



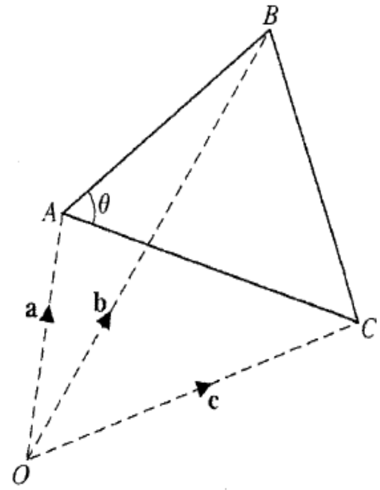
Now if the position vectors of A and B relative to O are $\mathbf{a}$ and $\mathbf{b}$, respectively, then

$$\begin{aligned}\text{area of } \triangle AOB &= \tfrac{1}{2}(OA)(OB)\sin\theta \\ &= \tfrac{1}{2}|\mathbf{a}|\,|\mathbf{b}|\sin\theta \\ &= \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}|\end{aligned}$$

- Area of $\triangle AOB = \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

If you have a triangle ABC where the position vectors of A, B, C relative to an origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, then the area of the triangle can be calculated in a similar fashion.

The area of the triangle is given by $\frac{1}{2}(AB)(AC)\sin\theta$, where $\theta = \angle BAC$.



That is, $\frac{1}{2}|\overrightarrow{AB}|\,|\overrightarrow{AC}|\sin\theta$

But $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$ so the area of the triangle is

$$\begin{aligned}&\tfrac{1}{2}|\mathbf{b}-\mathbf{a}|\,|\mathbf{c}-\mathbf{a}|\sin\theta \\ &= \tfrac{1}{2}|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})| \\ &= \tfrac{1}{2}|(\mathbf{b}\times\mathbf{c}) - (\mathbf{b}\times\mathbf{a}) - (\mathbf{a}\times\mathbf{c}) + (\mathbf{a}\times\mathbf{a})| \\ &= \tfrac{1}{2}|\mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a} + \mathbf{a}\times\mathbf{b}| \\ &= \tfrac{1}{2}|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|\end{aligned}$$

since $\mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c}$, $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ and $\mathbf{a}\times\mathbf{a} = \mathbf{0}$.

- Area of $\triangle ABC = \frac{1}{2}|\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{c} + \mathbf{c}\times\mathbf{a}|$

Example

Find the area of the triangle OAB where O is the origin, A has position vector $2\mathbf{i} - \mathbf{j} + \mathbf{k}$ and B has position vector $3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.

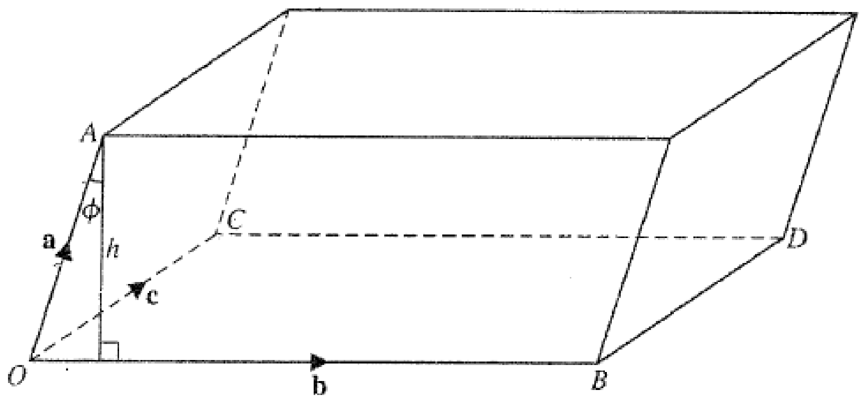
Solution

Example

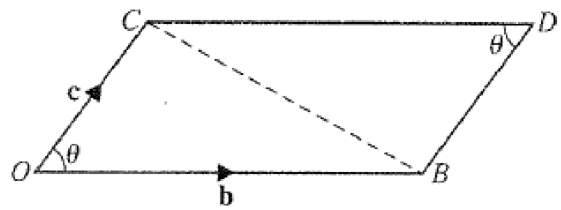
Find the area of the triangle ABC where the position vectors of A, B, C relative to the origin O are $2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$, $3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $-\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ respectively.

Solution

Volume of a parallelepiped



The volume of a parallelepiped is given by (area of base) × h where h is the perpendicular distance between the base and the top face. In the parallelepiped above, O is the origin and A, B, C have position vectors **a**, **b** and **c** respectively. The base of the parallelepiped is a parallelogram.



Now the area of $\triangle OBC$ is $\frac{1}{2}(OB)(OC)\sin\theta$ and the area of $\triangle DCB$ is $\frac{1}{2}(DC)(DB)\sin\theta$. (You should know that the opposite angles of a parallelogram are equal in size.) But because $OBDC$ is a parallelogram, $OC = BD$ and $OB = CD$.

So:

$$\begin{aligned}\text{area of parallelogram } OBDC &= \tfrac{1}{2}(OB)(OC)\sin\theta + \tfrac{1}{2}(DC)(DB)\sin\theta \\ &= \tfrac{1}{2}(OB)(OC)\sin\theta + \tfrac{1}{2}(OB)(OC)\sin\theta \\ &= (OB)(OC)\sin\theta \\ &= |\mathbf{b}\times\mathbf{c}|\end{aligned}$$

The volume of the parallelepiped is therefore $|\mathbf{b}\times\mathbf{c}|h$.

Now $$\frac{h}{OA} = \cos\phi$$

where ϕ is the angle between the vertical and OA.

So: $$h = OA \cos\phi$$

and the volume is $|\mathbf{b} \times \mathbf{c}| OA \cos\phi$

$$= |\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}| \cos\phi$$

$$= |\mathbf{a}|\,|\mathbf{b} \times \mathbf{c}| \cos\phi$$

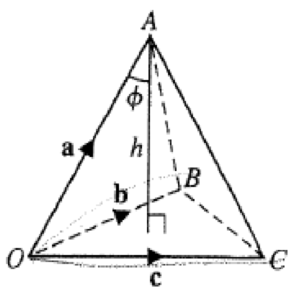
But $\mathbf{b} \times \mathbf{c}$ is vertically up, in the direction of h, since $\mathbf{b} \times \mathbf{c}$ is perpendicular to both **b** and **c**.

So ϕ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, and $|\mathbf{a}|\,|\mathbf{b} \times \mathbf{c}| \cos\phi$ is the scalar product of **a** and $\mathbf{b} \times \mathbf{c}$.

- Thus the volume of the parallelepiped is given by $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$, which is usually written without the brackets, because there can be no confusion, as simply $\mathbf{a}.\mathbf{b} \times \mathbf{c}$. This quantity is usually known as a **triple scalar product**.

Volume of a tetrahedron

The volume of a tetrahedron is given by the formula $\frac{1}{3}$ (area of base) $\times\, h$, where h is the perpendicular height.



In the tetrahedron $OABC$ above, you have O as the origin, **a** as the position vector of A, **b** as the position vector of B and **c** as the position vector of C. The perpendicular height makes an angle ϕ with OA.

You know that the area of the triangular base is $\frac{1}{2}|\mathbf{b} \times \mathbf{c}|$.

Also: $$h = OA \cos\phi = |\mathbf{a}| \cos\phi$$

So the volume of the tetrahedron is given by

$$\tfrac{1}{3} \times \tfrac{1}{2} |\mathbf{b} \times \mathbf{c}|\,|\mathbf{a}| \cos\phi$$

But because $\mathbf{b} \times \mathbf{c}$ is in the direction of h, the angle ϕ is the angle between $\mathbf{b} \times \mathbf{c}$ and **a**.

- Therefore the volume of the tetrahedron is given by

$$\tfrac{1}{6}\mathbf{a}.\mathbf{b} \times \mathbf{c}$$

Evaluating the triple scalar product

You know that $\mathbf{b} \times \mathbf{c}$ can be evaluated as

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

where $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$

So $\mathbf{b} \times \mathbf{c} = (b_2c_3 - b_3c_2)\,\mathbf{i} + (b_3c_1 - b_1c_3)\,\mathbf{j} + (b_1c_2 - b_2c_1)\,\mathbf{k}$ (A)

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then:

$$\mathbf{a}.\mathbf{b} \times \mathbf{c} = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \quad \text{(B)}$$

However, if you compare (A) and (B) you will see that they are the same, except that in (B) $\mathbf{i}$ is replaced by a_1, $\mathbf{j}$ is replaced by a_2 and $\mathbf{k}$ is replaced by a_3. This leads to the conclusion that

■ $$\mathbf{a}.\mathbf{b} \times \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

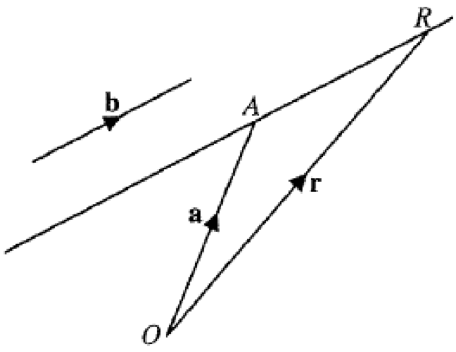
Example

A tetrahedron has vertices at $A(0, 1, 0)$, $B(1, 1, 2)$, $C(-2, 1, 3)$ and $D(2, 0, 1)$ relative to the origin O. Find the volume of the tetrahedron.

Solution

*P4 book page 220 Ex6B Q1,3,5-11,13,14,16

The equation of a straight line



If a line is parallel to a vector **b**, and if a point A on the line has position vector **a** and any other point R on the line has position vector **r**, then an equation of the line is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$$

where λ is a scalar parameter (Book P3, page 138).

Now the vector $\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$ and $\overrightarrow{AR}$ is parallel to **b**. But you learned earlier (page 211) that if two vectors are parallel, then their vector product is zero.

■ **So another form of the equation of the straight line is**

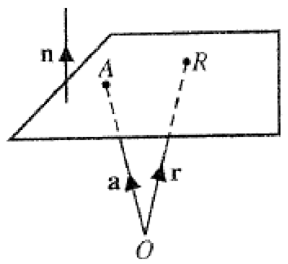
$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$$

Example

Find an equation of the straight line which passes through point $A(2, -1, 1)$ and is parallel to $\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$.

Solution

The scalar product form of the equation of a plane



The point A, with position vector $\mathbf{a}$ lies in a given plane. The vector $\mathbf{n}$ is perpendicular to the plane. The point R with position vector $\mathbf{r}$ is any other point in the plane.

Then: $$\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$$

Since both A and R lie in the given plane, then $\overrightarrow{AR}$ must lie in the plane. But if $\mathbf{n}$ is perpendicular to the plane then the directions of $\mathbf{n}$ and $\overrightarrow{AR} = \mathbf{r} - \mathbf{a}$ must be perpendicular. However, if two vectors are perpendicular then their scalar product is zero (see Book P3, page 132).

So: $$(\mathbf{r} - \mathbf{a}).\mathbf{n} = 0$$

or $$\mathbf{r}.\mathbf{n} - \mathbf{a}.\mathbf{n} = 0$$

$$\Rightarrow \quad \mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$$

If $\mathbf{a}.\mathbf{n} = p$ then an equation of the plane is $\mathbf{r}.\mathbf{n} = p$.

- Given that a plane contains a point with position vector $\mathbf{a}$, that $\mathbf{r}$ is the position vector of any other point in the plane and that $\mathbf{n}$ is perpendicular to the plane, the scalar product form of the equation of the plane is $\mathbf{r}.\mathbf{n} = p$, where $p = \mathbf{a}.\mathbf{n}$.

Cartesian equation of a Plane

Consider a plane with equation $r.n = \rho$ where $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Writing ***r***, the position vector of the general point as $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Gives $\begin{pmatrix} x \\ y \\ z \end{pmatrix}.\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \rho$

i.e. $ax + by + cz = \rho$

Thus a plane perpendicular to $ai + bj + ck$ has Cartesian equation $ax + by + cz = \rho$

Note

To show that a line lies in a given plane, show either:-

- Line and plane are parallel AND They have a common point
  OR
- The line and plane contain two common points

Example

The point A with position vector $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ lies in a plane. The vector $-\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to the plane. Find an equation of the plane

(a) in scalar product form

(b) in cartesian form.

Solution

Example

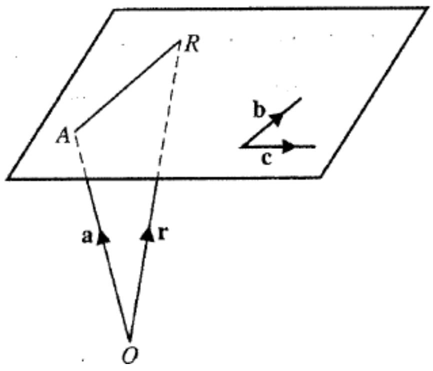
Show that the line with equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j})$ where λ is a scalar parameter lies in the plane with equation

r.(i-2j+2k)=-1.

Solution

The vector equation of a plane

If A lies in the given plane and has position vector $\mathbf{a}$ and R is any point in the plane and has position vector $\mathbf{r}$, and $\mathbf{b}$ and $\mathbf{c}$ are two non-parallel vectors in the plane, neither of which is zero, then



$$\mathbf{r} = \mathbf{a} + \overrightarrow{AR}$$

But $\overrightarrow{AR}$, since it lies in the plane, can be written:

$$\overrightarrow{AR} = \lambda\mathbf{b} + \mu\mathbf{c}$$

where λ, μ are scalar parameters.

So: $$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$$

- The vector equation of a plane where $\mathbf{a}$ is the position vector of a point in the plane and $\mathbf{b}$ and $\mathbf{c}$ are non-parallel vectors in the plane, neither of which is zero, is given by

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}, \quad \lambda, \mu \text{ scalars}$$

Example

Three points in a plane have coordinates $A(1, -1, 0)$, $B(0, 1, 1)$ and $C(2, 1, -2)$ referred to an origin O. Find a vector equation of the plane.

Solution

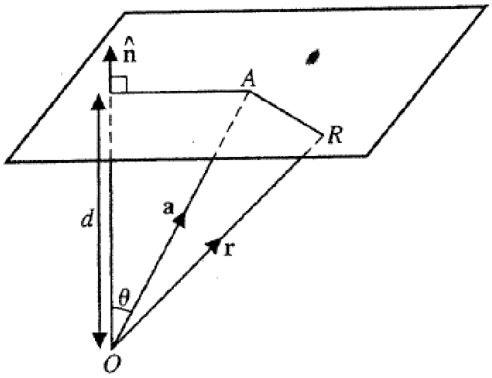
Example

Find a cartesian equation of the plane containing the points $A(1, 1, 1)$, $B(2, 1, 0)$ and $C(2, 2, -1)$.

Solution

*P4 book page 228 Ex6C Q1ab,2ab,3ab,4ab,5ab,6a,7ac,8ac,9-12

Distance of a point from a plane



The diagram shows a plane which contains the point A, with position vector $\mathbf{a}$, and also contains any other point R, with position vector $\mathbf{r}$. The unit vector $\hat{\mathbf{n}}$ is perpendicular to the plane. The line OA makes an angle θ with $\hat{\mathbf{n}}$ and d is the distance of O from the plane.

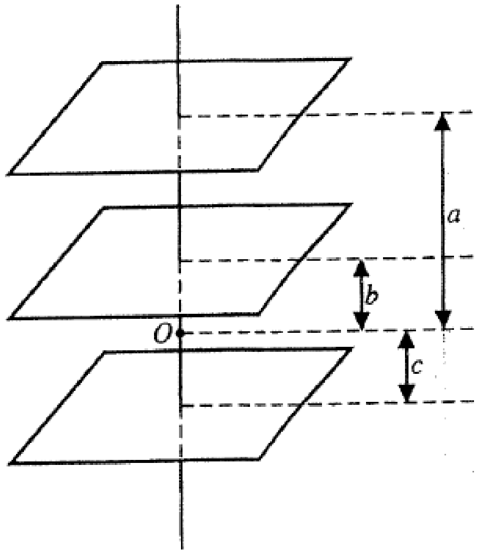
Then: $d = a\cos\theta$

$= a.1.\cos\theta$

$= |\mathbf{a}|\,|\hat{\mathbf{n}}|\cos\theta$, since $\hat{\mathbf{n}}$ is a unit vector

So $$d = \mathbf{a}.\hat{\mathbf{n}}$$

Now an equation of the plane is $\mathbf{r}.\mathbf{n} = \mathbf{a}.\mathbf{n}$. Thus if you replace the vector $\mathbf{n}$ by $\hat{\mathbf{n}}$ in the scalar product form of the equation of the plane you get $\mathbf{r}.\hat{\mathbf{n}} = d$ where d is the perpendicular distance of the origin from the plane.



Consider 3 parallel planes Π_1, Π_2 and Π_3 with vector equations

$\underset{\sim}{r}.\hat{\underset{\sim}{n}} = 3$ Π_1 will be a distance 3 units from the origin.

$\underset{\sim}{r}.\hat{\underset{\sim}{n}} = 1$ Π_2 " " 1 " ".

$\underset{\sim}{r}.\hat{\underset{\sim}{n}} = -2$ Π_3 " " 2 " ".

and Π_3 is on the other side of the origin from Π_1 and Π_2.

$\Rightarrow \Pi_1$ is 2 units from Π_2 and Π_1 is 5 units from Π_3.

Example

Find the perpendicular distance of the origin from the plane with equation $\mathbf{r}.(2\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}) = 7$.

Solution

Example

Find the distance of the point $(3, 1, 6)$ from the plane with equation $\mathbf{r}.(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 13$.

Solution

The line of intersection of two planes

In general when two planes intersect their intersection is a straight line. If you can write the equation of each plane in cartesian form, then by solving the two equations simultaneously you should be able to obtain an equation of the line of intersection, as the next example shows.

Example

Find, in vector form, an equation of the line of intersection of the plane $\mathbf{r}.(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 3$ with the plane $\mathbf{r}.(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 5$.

Solution

$\mathbf{r}.(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 3$ can be written:

$$x + y + z = 3 \qquad (1)$$

and $\mathbf{r}.(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = 5$ can be written:

$$x + 2y + 3z = 5 \qquad (2)$$

(2) – (1) gives $\quad y + 2z = 2$

So: $\quad y = 2 - 2z$

Substituting in (1) gives:

$$x + 2 - 2z + z = 3$$

$$\Rightarrow \quad x - z = 1$$

or $\quad x = 1 + z$

If you let $z = \lambda$, say, then

$$\frac{x-1}{1} = \frac{2-y}{2} = \frac{z}{1} (= \lambda)$$

which is an equation of the line of intersection in cartesian form.

So: $\quad x = 1 + \lambda,\ y = 2 - 2\lambda,\ z = \lambda$

If $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

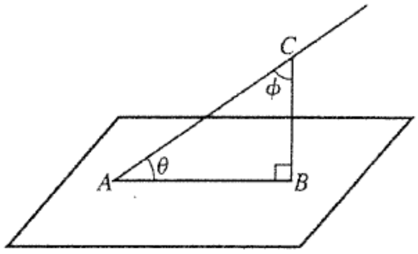
then: $\quad \mathbf{r} = (1+\lambda)\mathbf{i} + (2-2\lambda)\mathbf{j} + \lambda\mathbf{k}$

That is: $\quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

which is an equation of the line of intersection in vector form.

The angle between a line and a plane

The angle between a line and a plane is the angle between the line and its projection on the plane (see Book P2, page 188). In the diagram, AC is the line and AB is its projection on the plane. So you have to find the angle $CAB = \theta$.



But $\theta = 90 - \emptyset$ is the angle between the line BC i.e. $\emptyset$ is the angle between AC and a normal to the plane.

*So, if asked to find the angle between a line and a plane, first find the angle between the line and the normal to the plane and then subtract the answer from 90°. This is the required angle.

<u>Example</u>

Find the acute angle between the line with equations

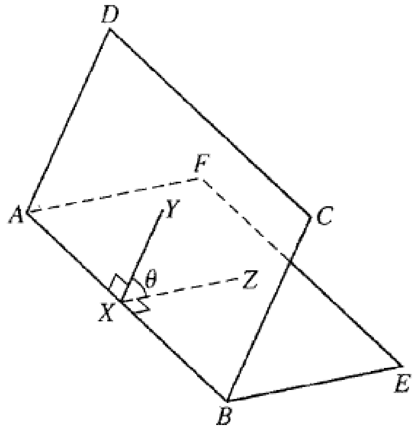
$$\frac{x+1}{2} = y - 2 = \frac{z-3}{-2} = \lambda$$

and the plane with equation

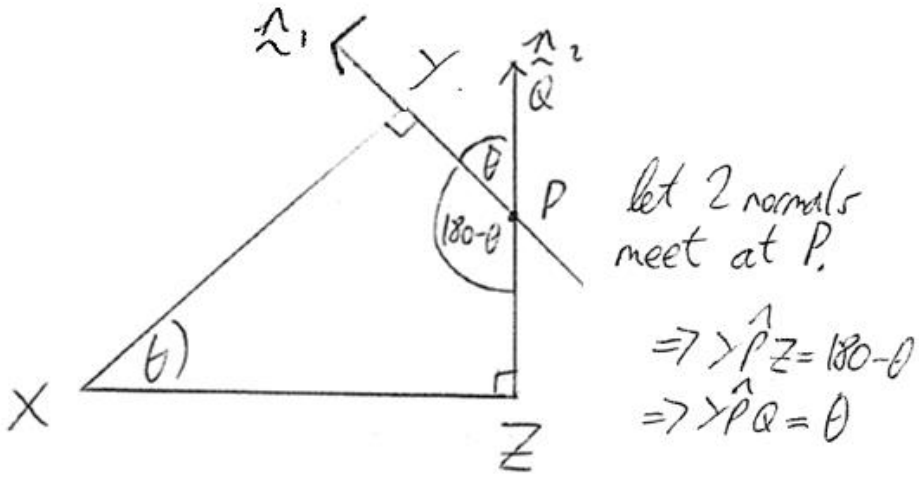
$$2x + 3y - 7z = 5$$

<u>Solution</u>

Angle between two planes



The angle between two planes is the angle between two perpendiculars (one in each plane) drawn from a point on their line of intersection (Book P2, page 191). If you are given vector equations of the two planes you need to be able to calculate $\angle YXZ = \theta$. So from Y you draw the perpendicular to the plane $ABCD$ and from Z you draw the perpendicular to the plane $ABEF$.



Result:- The angle between two planes is the angle between the two normal.

Example Find the angle between the planes with equations

$$\mathbf{r}.(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = 5$$

and

$$\mathbf{r}.(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = 7$$

Solution

*P4 book page 236 Ex6D Q1ac,2ac,3,4ab,5ab,6,7,8ab,11,13-17,19,20,21