Roots of Polynomials

A quadratic equation of the form $ax^2 + bx + c = 0$, $x \in \mathbb{C}$, where a, b and c are real constants, can have two real roots, one repeated (real) root or two complex roots.

If the roots of this equation are α and β, you can determine the relationship between the **coefficients** of the terms in the quadratic equation and the values of α and β:

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$
$$= a(x^2 - \alpha x - \beta x + \alpha\beta)$$
$$= ax^2 - a(\alpha + \beta)x + a\alpha\beta$$

Write the quadratic expression in factorised form, then rearrange into the form $ax^2 + bx + c$.

So $b = -a(\alpha + \beta)$ and $c = a\alpha\beta$.

- **If α and β are roots of the equation $ax^2 + bx + c = 0$, then:**
  - $\alpha + \beta = -\frac{b}{a}$
  - $\alpha\beta = \frac{c}{a}$

sum of roots $= -\frac{b}{a}$

Product of roots $= \frac{c}{a}$

eg $x^2 - 7x + 12 = 0$
$(x-3)(x-4) = 0$
$x = 3 \quad x = 4$
so $\alpha = 3 \quad \beta = 4$
$\alpha + \beta = 7$ and $\frac{-b}{a} = \frac{-(-7)}{1} = 7$
$\alpha + \beta = -\frac{b}{a}$
$\alpha\beta = 3 \times 4 = 12$ and $\frac{c}{a} = \frac{12}{1} = 12$

Note The sum of the roots is $-\frac{b}{a}$ and the product of the roots is $\frac{c}{a}$. Note that these values are real even if the roots are complex, because the sum or product of a conjugate pair is real.

Example 1

The roots of the quadratic equation $2x^2 - 5x - 4 = 0$ are α and β. Without solving the equation, find the values of:

a $\alpha + \beta$ **b** $\alpha\beta$ **c** $\frac{1}{\alpha} + \frac{1}{\beta}$ **d** $\alpha^2 + \beta^2$

(a) $a = 2$
$b = -5$
$c = -4$
$\alpha + \beta = -\frac{b}{a}$
$= \frac{5}{2}$

(b) $\alpha\beta = \frac{c}{a}$
$= \frac{-4}{2}$
$= -2$

(c) $\frac{1}{\alpha} + \frac{1}{\beta} = ?$
$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$
$= \frac{(\frac{5}{2})}{-2}$
$= \frac{-5}{4}$

(d) $\alpha^2 + \beta^2 = ?$
$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
$= (\frac{5}{2})^2 - 2(-2)$
$= \frac{25}{4} + 4 = 10\frac{1}{4}$

Problem-solving

Write each expression in terms of $\alpha + \beta$ and $\alpha\beta$:
$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta \Rightarrow \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

Example 2

The roots of a quadratic equation $ax^2 + bx + c = 0$ are $\alpha = -\frac{3}{2}$ and $\beta = \frac{5}{4}$
Find integer values for a, b and c.

$\alpha = -\frac{3}{2}$

$\beta = \frac{5}{4}$

$\alpha + \beta = -\frac{3}{2} + \frac{5}{4}$

$= -\frac{6}{4} + \frac{5}{4}$

$= -\frac{1}{4}$

$\Rightarrow -\frac{b}{a} = -\frac{1}{4}$

$\alpha\beta = -\frac{3}{2} \times \frac{5}{4}$

$= -\frac{15}{8}$

$\Rightarrow \frac{c}{a} = -\frac{15}{8}$

put $a = 1$

$\Rightarrow -b = -\frac{1}{4}$

$b = \frac{1}{4}$

and $c = -\frac{15}{8}$

$x^2 + \frac{1}{4}x - \frac{15}{8} = 0$

$\times 8$

$8x^2 + 2x - 15 = 0$

*Exercise 4A Q 1, 4, 6

Roots of a cubic equation

A cubic equation of the form $ax^3 + bx^2 + cx + d = 0$, $x \in \mathbb{C}$, where a, b, c and d are real constants, will always have at least one real root. It will also have either two further real roots, one further repeated (real) root or two complex roots.

If the roots of this equation are α, β and γ, you can determine the relationship between the coefficients of the terms in the cubic equation and the values of α, β and γ:

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a(x-\alpha)(x-\beta)(x-\gamma) \\ &= a(x^3 - \alpha x^2 - \beta x^2 - \gamma x^2 + \alpha\beta x + \beta\gamma x + \gamma\alpha x - \alpha\beta\gamma) \\ &= ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\beta\gamma+\gamma\alpha)x - a\alpha\beta\gamma \end{aligned}$$

So $b = -a(\alpha+\beta+\gamma)$, $c = a(\alpha\beta+\beta\gamma+\gamma\alpha)$ and $d = -a\alpha\beta\gamma$.

- **If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:**
  - $\alpha + \beta + \gamma = -\frac{b}{a}$ ← sum of roots
  - $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ ← sum of pairs
  - $\alpha\beta\gamma = -\frac{d}{a}$ ← products

Note As with the rule for quadratic equations, the sum of the roots is $-\frac{b}{a}$, and the sum of the products of all possible pairs of roots is $\frac{c}{a}$

Example 3

α, β and γ are the roots of the cubic equation $2x^3 + 3x^2 - 4x + 2 = 0$. Without solving the equation, find the values of:

a $\alpha + \beta + \gamma$ **b** $\alpha\beta + \beta\gamma + \gamma\alpha$ **c** $\alpha\beta\gamma$ **d** $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$a=2$
$b=3$
$c=-4$
$d=2$

(a) $\alpha+\beta+\gamma = \frac{-b}{a}$
$= \frac{-3}{2}$

(b) $\alpha\beta+\beta\gamma+\gamma\alpha$
$= \frac{c}{a}$
$= \frac{-4}{2}$
$= -2$

(c) $\alpha\beta\gamma$
$= \frac{-d}{a}$
$= \frac{-2}{2}$
$= -1$

(d) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
$= \frac{\beta\gamma + \alpha\gamma + \beta\alpha}{\alpha\beta\gamma}$
$= \frac{-2}{-1}$
$= 2$

Example 4

The roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$ are $\alpha = 1 - 2i$, $\beta = 1 + 2i$ and $\gamma = 2$. Find integer values for a, b, c and d.

$\alpha+\beta+\gamma = 1-2i + 1+2i + 2$
$= 4$

$\frac{-b}{a} = 4$ *

$\alpha\beta + \alpha\gamma + \beta\gamma = (1-2i)(1+2i)$
$+ 2(1-2i)$
$+ 2(1+2i)$
$= ((1)^2 - (2i)^2) + (2-4i) + (2+4i)$
$= (1-4(-1)) + 4$
$= (1+4)+4 = 5+4 = 9$

$(i^2 = -1)$

$\Rightarrow \frac{c}{a} = 9$ *

$\alpha\beta\gamma = (1-2i)(1+2i)(2)$
$= (5)(2)$
$= 10 \Rightarrow \frac{-d}{a} = 10$ *

Let $a=1$
$\Rightarrow -b=4$ $b=-4$
and $c=9$
and $-d=10$ $d=-10$

*Exercise 4B Q1, 4

Roots of a quartic equation

Consider the quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, $x \in \mathbb{C}$, where a, b, c, d and e are real numbers. If the roots of the equation are α, β, γ and δ, you can determine the relationship between the coefficients of the terms in the equation and the values of α, β, γ and δ:

$$\begin{aligned} ax^4 + bx^3 + cx^2 + dx + e &= a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) \\ &= a(x^4 - \alpha x^3 - \beta x^3 - \gamma x^3 - \delta x^3 + \alpha\beta x^2 + \beta\gamma x^2 + \gamma\alpha x^2 + \gamma\delta x^2 + \alpha\delta x^2 \\ &\quad + \beta\delta x^2 - \alpha\beta\gamma x - \alpha\beta\delta x - \alpha\gamma\delta x - \beta\gamma\delta x + \alpha\beta\gamma\delta) \\ &= ax^4 - a(\alpha+\beta+\gamma+\delta)x^3 + a(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)x^2 \\ &\quad - a(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)x + a\alpha\beta\gamma\delta \end{aligned}$$

So $b = -a(\alpha+\beta+\gamma+\delta)$, $c = a(\alpha\beta+\beta\gamma+\gamma\alpha+\gamma\delta+\alpha\delta+\beta\delta)$, $d = -a(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)$ and $e = a\alpha\beta\gamma\delta$.

- **If α, β, γ and δ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then:**
  - $\alpha+\beta+\gamma+\delta = -\frac{b}{a}$ ← sum of roots
  - $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}$ ← sum of pairs
  - $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a}$ ← sum of triples
  - $\alpha\beta\gamma\delta = \frac{e}{a}$ ← products

Notation You can use the following abbreviations for these results in your working:

$\Sigma\alpha = -\frac{b}{a}$ $\quad \Sigma\alpha\beta = \frac{c}{a}$ $\quad \Sigma\alpha\beta\gamma = -\frac{d}{a}$

Example 5

The equation $x^4 + 2x^3 + px^2 + qx - 60 = 0$, $x \in \mathbb{C}$, $p, q \in \mathbb{R}$, has roots α, β, γ and δ.
Given that $\gamma = -2 + 4i$ and $\delta = \gamma^*$, γ^* means the conjugate of γ ie, $\delta = -2-4i$

a show that $\alpha + \beta - 2 = 0$ and that $\alpha\beta + 3 = 0$.

b Hence find all the roots of the quartic equation and find the values of p and q.

(a) $\alpha+\beta+\gamma+\delta = \frac{-b}{a}$

$\alpha+\beta+(-2+4i)+(-2-4i) = -2$

$\alpha+\beta-4 = -2$

$\therefore \quad \alpha+\beta-2=0$

$\alpha\beta\gamma\delta = \frac{e}{a}$

$\alpha\beta(-2+4i)(-2-4i) = -60$

$\alpha\beta((-2)^2-(4i)^2) = -60$

$(4i)^2 = 16(-1) = -16$

$\alpha\beta(20) = -60$

$\alpha\beta = \frac{-60}{20}$

$\alpha\beta = -3 \rightarrow \alpha\beta+3=0$

(b) $x^4+2x^3+px^2+qx-60=0$

$\alpha+\beta = 2$ ... (1)

$\alpha\beta = -3$ ... (2)

$\beta = 2-\alpha$ ... (1)

sub into (2)

$\alpha(2-\alpha) = -3$

$2\alpha - \alpha^2 = -3$

$0 = \alpha^2 - 2\alpha - 3$

$0 = (\alpha-3)(\alpha+1)$

$\alpha = 3 \qquad \alpha = -1$

when $\alpha = 3$: $\beta = 2-3$, $\beta = -1$

when $\alpha = -1$: $\beta = 2-(-1) = 3$

ans: $\alpha = -1, \beta = 3, \gamma = -2+4i, \delta = -2-4i$

$\alpha=-1, \beta=3, \gamma=-2+4i, \delta=-2-4i$

$\Sigma\alpha\beta = \frac{c}{a}$

$(-1)(3)+(-1)(-2+4i)+(-1)(-2-4i)+(3)(-2+4i)+(3)(-2-4i)$
$+(-2+4i)(-2-4i) = p$

$-3+2-4i+2+4i-6+12i-6-12i+(4+16) = p$

$p = 9$

$\Sigma\alpha\beta\gamma = \frac{-d}{a}$

$(-1)(3)(-2+4i)+(-1)(3)(-2-4i)+(-1)(-2+4i)(-2-4i)$
$+(3)(-2+4i)(-2-4i) = -q$

$6-12i+6+12i-(4+16)+3(4+16) = -q$

$-q = 52$

$q = -52$

*Exercise 4C Q 1, 5, 11

Expressions relating to the roots of a polynomial

You have already seen several results for finding the values of expressions relating to the roots of a polynomial.

- **The rules for reciprocals:**
  - **Quadratic:** $\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}$
  - **Cubic:** $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}$
  - **Quartic:** $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}=\frac{\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta}{\alpha\beta\gamma\delta}$
- **The rules for products of powers:**
  - **Quadratic:** $\alpha^n\times\beta^n=(\alpha\beta)^n$
  - **Cubic:** $\alpha^n\times\beta^n\times\gamma^n=(\alpha\beta\gamma)^n$
  - **Quartic:** $\alpha^n\times\beta^n\times\gamma^n\times\delta^n=(\alpha\beta\gamma\delta)^n$

In addition to these you have also used the following results for the roots of quadratic equations:

- $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
- $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

There are equivalent results to these for the roots of cubic and quartic equations.

Example 6

a Expand $(\alpha+\beta+\gamma)^2$.

b A cubic equation has roots α, β, γ such that $\alpha\beta+\beta\gamma+\gamma\alpha=7$ and $\alpha+\beta+\gamma=-3$. Find the value of $\alpha^2+\beta^2+\gamma^2$.

(a) $(\alpha+\beta+\gamma)^2=(\alpha+\beta+\gamma)(\alpha+\beta+\gamma)$
$=\alpha^2+\alpha\beta+\alpha\gamma+\beta\alpha+\beta^2+\beta\gamma$
$+\gamma\alpha+\gamma\beta+\gamma^2$
$=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)$

(b) $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$
$=(-3)^2-2(7)$
$=9-14$
$=-5$

You can find an expression for the sum of the squares of a quartic equation in a similar way, by multiplying out $(\alpha+\beta+\gamma+\delta)^2$.

- **The rules for sums of squares:**
  - **Quadratic:** $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
  - **Cubic:** $\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$
  - **Quartic:** $\alpha^2+\beta^2+\gamma^2+\delta^2=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

Note If you learn these you can use them without proof in your exam.

You can find a similar result for the sum of the cubes of a cubic equation by multiplying out $(\alpha+\beta+\gamma)^3$.

- **The rules for sums of cubes:**
  - **Quadratic:** $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$
  - **Cubic:** $\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)+3\alpha\beta\gamma$

Note The result for the sum of cubes for a quartic equation is not required.

*Exercise 4D Q1, 4, 7

Linear transformations of roots

Given the sums and products of the roots of a polynomial, it is possible to find the equation of a second polynomial whose roots are a linear transformation of the roots of the first.

For example, if the roots of a cubic equation are α, β and γ, you need to be able to find the equation of a polynomial with roots $(\alpha + 2)$, $(\beta + 2)$ and $(\gamma + 2)$, or 3α, 3β and 3γ.

Example 8

The cubic equation $x^3 - 2x^2 + 3x - 4 = 0$ has roots α, β and γ.
Find the equations of the polynomials with roots:

a 2α, 2β and 2γ **b** $(\alpha + 3)$, $(\beta + 3)$ and $(\gamma + 3)$

Problem-solving

Find the sum $\Sigma\alpha$, the pair sum $\Sigma\alpha\beta$ and the product $\alpha\beta\gamma$ for the original equation. Then use these values to find the equivalent sums and products for an equation with roots 2α, 2β and 2γ.

(a) $x^3 - 2x^2 + 3x - 4 = 0$
has roots α, β, γ
$\alpha + \beta + \gamma = 2 \leftarrow -\frac{b}{a}$
$\alpha\beta + \alpha\gamma + \beta\gamma = 3 \leftarrow \frac{c}{a}$
$\alpha\beta\gamma = 4 \leftarrow -\frac{d}{a}$
If roots are
$2\alpha, 2\beta, 2\gamma$
$2\alpha + 2\beta + 2\gamma = 2(2)$
$= 4 \leftarrow -\frac{b}{a}$
$\therefore -b = 4 \quad a = 1$
$b = -4$
pairs $(2\alpha)(2\beta) + (2\alpha)(2\gamma) + (2\beta)(2\gamma) =$
$4(\alpha\beta + \alpha\gamma + \beta\gamma) = 12$
$\therefore \frac{c}{a} = 12 \Rightarrow c = 12$

$(2\alpha)(2\beta)(2\gamma) = 8\alpha\beta\gamma$
$= 32$
$\therefore -\frac{d}{a} = 32 \quad -d = 32$
$d = -32$
equation is
$x^3 - 4x^2 + 12x - 32 = 0$

Method 2
let $w = 2x$
$\therefore x = \frac{w}{2}$
sub into $x^3 - 2x^2 + 3x - 4 = 0$
$\left(\frac{w}{2}\right)^3 - 2\left(\frac{w}{2}\right)^2 + 3\left(\frac{w}{2}\right) - 4 = 0$
$\frac{w^3}{8} - \frac{w^2}{2} + \frac{3w}{2} - 4 = 0 \quad \times 8$
$w^3 - 4w^2 + 12w - 32 = 0$
or $x^3 - 4x^2 + 12x - 32 = 0$

(b) $x^3 - 2x^2 + 3x - 4 = 0 \quad \ldots (1)$
Sum $(\alpha+3) + (\beta+3) + (\gamma+3)$
$= \alpha + \beta + \gamma + 9$
$= 11$
$\therefore -\frac{b}{a} = 11$ let $a = 1$ $\therefore -b = 11$
$b = -11$
Pairs $(\alpha+3)(\beta+3) + (\alpha+3)(\gamma+3) + (\beta+3)(\gamma+3)$
$= \alpha\beta + 3\alpha + 3\beta + 9 + \alpha\gamma + 3\alpha + 3\gamma + 9 + \beta\gamma + 3\beta + 3\gamma + 9$
$= \alpha\beta + \alpha\gamma + \beta\gamma + 6(\alpha + \beta + \gamma) + 27$
$= 3 + 6(2) + 27$
$= 42$
$\therefore \frac{c}{a} = 42 \quad \therefore c = 42$

products
$(\alpha+3)(\beta+3)(\gamma+3)$
$= \alpha\beta\gamma + 3(\alpha\beta + \alpha\gamma + \beta\gamma)$
$+ 9(\alpha + \beta + \gamma) + 27$
$= 4 + 3(3) + 9(2) + 27$
$= 58$
$-\frac{d}{a} = 58 \quad \therefore -d = 58$
$d = -58$
equation is
$x^3 - 11x^2 + 42x - 58 = 0$

Method 2
let $w = x + 3$
$x = w - 3$
sub into (1)
$(w-3)^3 - 2(w-3)^2 + 3(w-3) - 4 = 0$
$\vdots$
$w^3 - 11w^2 + 42w - 58 = 0$
or
$x^3 - 11x^2 + 42x - 58 = 0$

Example 9

The quartic equation $x^4 - 3x^3 + 15x + 1 = 0$ has roots α, β, γ and δ. Find the equation with roots $(2\alpha + 1)$, $(2\beta + 1)$, $(2\gamma + 1)$ and $(2\delta + 1)$.

let $w = 2x + 1$

$x = \frac{w-1}{2}$

sub into $x^4 - 3x^3 + 15x + 1 = 0$

$\left(\frac{w-1}{2}\right)^4 - 3\left(\frac{w-1}{2}\right)^3 + 15\left(\frac{w-1}{2}\right) + 1 = 0$

$\frac{1}{16}(w^4 - 4w^3 + 6w^2 - 4w + 1) - \frac{3}{8}(w^3 - 3w^2 + 3w - 1) + \frac{15}{2}(w-1) + 1 = 0$

$w^4 - 4w^3 + 6w^2 - 4w + 1 - 6(w^3 - 3w^2 + 3w - 1) + 120(w-1) + 16 = 0$ $\times 16$

$w^4 - 4w^3 + 6w^2 - 4w + 1 - 6w^3 + 18w^2 - 18w + 6 + 120w - 120 + 16 = 0$

$w^4 - 10w^3 + 24w^2 + 98w - 97 = 0$

or $x^4 - 10x^3 + 24x^2 + 98x - 97 = 0$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

*Exercise 4E Q1, 3, 5, 7

*Mixed Exercise Q1, 2, 4, 8, 10, 11

Summary of key points

1 If α and β are roots of the equation $ax^2 + bx + c = 0$, then:

- $\alpha + \beta = -\frac{b}{a}$
- $\alpha\beta = \frac{c}{a}$

2 If α, β and γ are roots of the equation $ax^3 + bx^2 + cx + d = 0$, then:

- $\alpha + \beta + \gamma = \Sigma\alpha = -\frac{b}{a}$
- $\alpha\beta + \beta\gamma + \gamma\alpha = \Sigma\alpha\beta = \frac{c}{a}$
- $\alpha\beta\gamma = -\frac{d}{a}$

3 If α, β, γ and δ are roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, then:

- $\alpha + \beta + \gamma + \delta = \Sigma\alpha = -\frac{b}{a}$
- $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \Sigma\alpha\beta = \frac{c}{a}$
- $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = \Sigma\alpha\beta\gamma = -\frac{d}{a}$
- $\alpha\beta\gamma\delta = \frac{e}{a}$

4 The rules for **reciprocals**:

- Quadratic: $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$
- Cubic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$
- Quartic: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} = \frac{\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta}{\alpha\beta\gamma\delta}$

5 The rules for **products of powers**:

- Quadratic: $\alpha^n \times \beta^n = (\alpha\beta)^n$
- Cubic: $\alpha^n \times \beta^n \times \gamma^n = (\alpha\beta\gamma)^n$
- Quartic: $\alpha^n \times \beta^n \times \gamma^n \times \delta^n = (\alpha\beta\gamma\delta)^n$

6 The rules for **sums of squares**:

- Quadratic: $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- Cubic: $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$
- Quartic: $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

7 The rules for **sums of cubes**:

- Quadratic: $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
- Cubic: $\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) + 3\alpha\beta\gamma$